## CONTROL SYSTEMS ENGINEERING D227

## SOLUTIONS Q6 2003

A system has an open loop transfer function $G(s)=\frac{K}{s(\tau s+1)}$ and it is used in a closed loop with unity feed back. A low frequency square wave input is applied and with gain K set to 10 and the system output indicates a maximum overshoot of $40 \%$ and requires 0.33 seconds to reach the peak value.
(a) Plot the open loop transfer function on a Nichols chart and establish the gain margin to limit the closed loop magnitude ratio to 4 db .
(b) Sketch the form of the closed loop frequency response on semi log paper.

Note - I spent a lot of time simulating this system with a computer and found there is no way that the specified system will produce an overshoot of $40 \%$ in 0.33 seconds no matter what the size of the step. With a gain of 20 it is possible to obtain a perfect match with the data and with the response to a standard second order system. Either I am totally missing the point or the question is unsolvable. I have taken $\mathrm{k}=$ 20 in order to produce the following solution.

## SOLUTION (a)

I believe the solution depends on knowing that the closed loop response to a step is a decaying oscillation similar to a standard second order system response. The low frequency square wave is interpreted as a step input of unknown settling level. First find the closed loop transfer function.
$\mathrm{G}(\mathrm{ol})=\frac{20}{\left(\tau \mathrm{~s}^{2}+\mathrm{s}\right)} \quad \mathrm{G}(\mathrm{cl})=\frac{\mathrm{G}(\mathrm{ol})}{1+\mathrm{G}(\mathrm{ol})}=\frac{1}{1 / \mathrm{G}(\mathrm{ol})+1}$
$\mathrm{G}(\mathrm{cl})=\frac{1}{\frac{\tau \mathrm{~s}^{2}+\mathrm{s}}{20}+1}=\frac{1}{0.05 \tau \mathrm{~s}^{2}+0.05 \mathrm{~s}+1}$
The standard second order transfer function is $\mathrm{G}(\mathrm{cl})=\frac{1}{\mathrm{~T}^{2} \mathrm{~s}^{2}+2 \delta \mathrm{Ts}+1}$
From the rubric of the exam paper we are given the following information for a unit step response for a standard second order system. I believe the data holds true for a step of any size.

Time to reach peak value. $=\mathrm{T}_{\mathrm{p}}$
$\mathrm{T}_{\mathrm{p}}=\frac{\pi}{\omega_{\mathrm{n}} \sqrt{1-\xi^{2}}}=$ time taken to reach the first peak $=0.33$
$\omega_{\mathrm{n}}=$ natural frequency $=1 / \mathrm{T} \zeta \equiv \delta$ (damping ratio)
The peak value is $\mathrm{M}_{\mathrm{pt}}$
$\mathrm{Mp}_{\mathrm{t}}=1+\mathrm{e}^{-\xi \pi / \sqrt{1-\zeta^{2}}}=$ peak magnitude of the first overshoot $=1.4$
$1.4=1+\mathrm{e}^{-\xi \pi / \sqrt{1-\zeta^{2}}} \quad 0.4=\mathrm{e}^{-\xi \pi / \sqrt{1-\zeta^{2}}} \quad \ln (0.4)=-0.9162=-\xi \pi / \sqrt{1-\zeta^{2}}$
$0.8396=\frac{(\xi \pi)^{2}}{1-\zeta^{2}} \quad 0.085068\left(1-\zeta^{2}\right)=\zeta^{2} \quad 0.085068=1.0850681 \zeta^{2}$
$\zeta^{2}=0.078399 \quad \zeta=0.28$
$0.33=\frac{\pi}{\omega_{\mathrm{n}} \sqrt{1-0.28^{2}}}$
$0.33 \omega_{\mathrm{n}}=3.272 \quad \omega_{\mathrm{n}}=9.917 \mathrm{rad} / \mathrm{s}$
$\mathrm{T}=1 / 9.917=0.1008$

$$
\mathrm{T}^{2}=0.05 \tau=0.01016 \tau=0.203
$$

(The computer simulation of both transfer functions yields near identical response for any step level)
Now we can produce data for the Nichols chart.
Evaluate the gain and phase angles for $G_{1}=\frac{20}{s} \quad G_{2}=\frac{1}{(0.203 s+1)}$
A suitable frequency range is 1 to $20 \mathrm{rad} / \mathrm{s}$.
The gain and phase angle for a polynomial of the form $\frac{K}{n+T s}$ is given by:-
$\mathrm{G}=\frac{\mathrm{K}}{\sqrt{\mathrm{n}^{2}+\mathrm{T}^{2} \omega^{2}}}$ and $\phi=-\tan ^{-1}\left(\frac{\omega \mathrm{~T}}{\mathrm{n}}\right)$ (Worth remembering)
Start with $\mathrm{G}_{1}=20 / \mathrm{s}$

$$
\begin{aligned}
& \mathrm{G}_{1}(\mathrm{~s})=\frac{20}{\mathrm{~s}} \quad \mathrm{n}=0 \quad \mathrm{~T}=1 \quad \mathrm{~K}=20 \\
& \mathrm{G}_{1}=\frac{20}{\sqrt{0+\omega^{2}}}=\frac{20}{\omega} \text { and } \phi=-\tan ^{-1}\left(\frac{\omega \mathrm{~T}}{0}\right)=-90^{\circ} \text { for all } \omega
\end{aligned}
$$

Next G ${ }_{2}=1 /(.203 \mathrm{~s}+1)$

$$
\begin{aligned}
& \mathrm{G}_{2}(\mathrm{~s})=\frac{1}{0.203 \mathrm{~s}+1} \quad \mathrm{n}=1 \quad \mathrm{~T}=0.203 \mathrm{~K}=1 \\
& \mathrm{G}_{2}=\frac{1}{\sqrt{1+0.203^{2} \omega^{2}}} \text { and } \phi=-\tan ^{-1}(\omega) \\
& \left|\mathrm{G}_{\text {ol }}\right|=\frac{20}{\omega} \mathrm{x} \frac{1}{\sqrt{1+0.203^{2} \omega^{2}}} \quad\left|\mathrm{G}_{\text {ol }}\right|(d b)=20 \log \left[\frac{20}{\omega} \mathrm{x} \frac{1}{\sqrt{1+0.203^{2} \omega^{2}}}\right]
\end{aligned}
$$

and $\varphi=-90^{\circ}-\tan ^{-1} \omega$
Evaluate and plot and we get the following result.


| $\omega$ | G db | $\phi$ |
| :---: | :---: | :---: |
| 1 | 25.845 | -101.475 |
| 2 | 19.337 | -112.097 |
| 3 | 15.108 | -121.341 |
| 4 | 11.78 | -129.076 |
|  | 8.966 | -135.426 |
| 6 | 6.507 | -140.613 |
| 7 | 4.32 | -144.865 |
| 8 | 2.351 | -148. |
| 9 | 0. | -151.306 |
| 10 | -1.073 | -1 |
| 1 | -2.579 | -155.876 |
| 12 | -3.973 | -157.681 |
| 13 | -5.27 | -15 |
| 14 | -6.481 | -160 |
| 15 | -7.618 | -161.819 |
| 16 | -8.687 | -162.887 |
| 17 | -9.697 | -163.84 |
| 18 | -10.654 | -164.694 |
| 19 | -11.562 | -165.465 |
|  | - 12.426 | -166.163 |

Phase Angle Degrees
In order to limit the peak closed loop gain to 4 db the plot must be moved down by 4 db . This is the change in the gain margin required.

## SOLUTION (b)

The simplest way to get the closed loop frequency response is by picking off the values of the closed loop gain (the contours) at relevant frequencies. This gives the following result. The result is shown plotted on semi $\log$ paper with the predicted result from a computer simulation of $\mathrm{G}(\mathrm{cl})=\frac{1}{\mathrm{~T}^{2} \mathrm{~s}^{2}+2 \delta \mathrm{Ts}+1}$
With $\mathrm{T}=0.1008\left(\omega_{\mathrm{n}}=9.917\right)$ and $\delta=0.28$. A good match is found. With a better Nichols chart than the one used, more points could be plotted. $\mathrm{G}(\mathrm{cl})=\frac{1}{\mathrm{~T}^{2} \mathrm{~s}^{2}+2 \delta \mathrm{Ts}+1}$


