CONTROL SYSTEMS ENGINEERING D227
SOLUTIONS Q2 2003

COMMENT- This seems an extremely long and arduous question. There must be an easier way but I haven't found it.

A system has an open loop transfer function $G(s) = \dfrac{2}{s(s+1)(s+2)}$ and it is used in a closed loop with unity feed back.

(a) Verify that the system is stable.
(b) Plot the CLOSED LOOP frequency response and show that the system may be approximated to a standard second order system.
(c) Determine the damping ratio and natural frequency of the approximated second order system and find the ±2% settling time of a unit step response.

(a) **SOLUTION**

There are several ways to tackle the problem. One would be to evaluate and plot the closed loop gain directly but this involves harder maths. Another method would be to plot the open loop gain on a polar plot and then calculate the closed loop gain from that. I have chosen to use the Nichol's chart and pick off the closed loop gain from that. An alternative method is also included.

STABILITY- A Nyquist Plot would do this but let's use the Routh Hurwitz criterion. First multiply out. $G(s) = 2/(s^3 + 3s^2 + 2s)$

The closed loop transfer function for unity feed back is $G_{cl} = 1/\{G(s) + 1\}$

Characteristic equation is $\dfrac{2}{s^3 + 3s^2 + 2s} + 1 = 0$ hence $s^3 + 3s^2 + 2s + 2 = 0$

If the highest power of the characteristic equation is 3 the criteria may be simplified as follows.

Characteristic equation = $(as^3 + bs^2 + cs + d)$ so $a = 1$ $b = 3$ $c = 2$ $d = 2$

If the next coefficient is negative the system is unstable and this is given by $R = c - ad/b$

$R = 2 - (1 \times 2/3) = 2 - 2/3 = 1.33$ This is positive so the system is stable.

(b) **SOLUTION**

CLOSED LOOP GAIN USING THE NICHOLS CHART

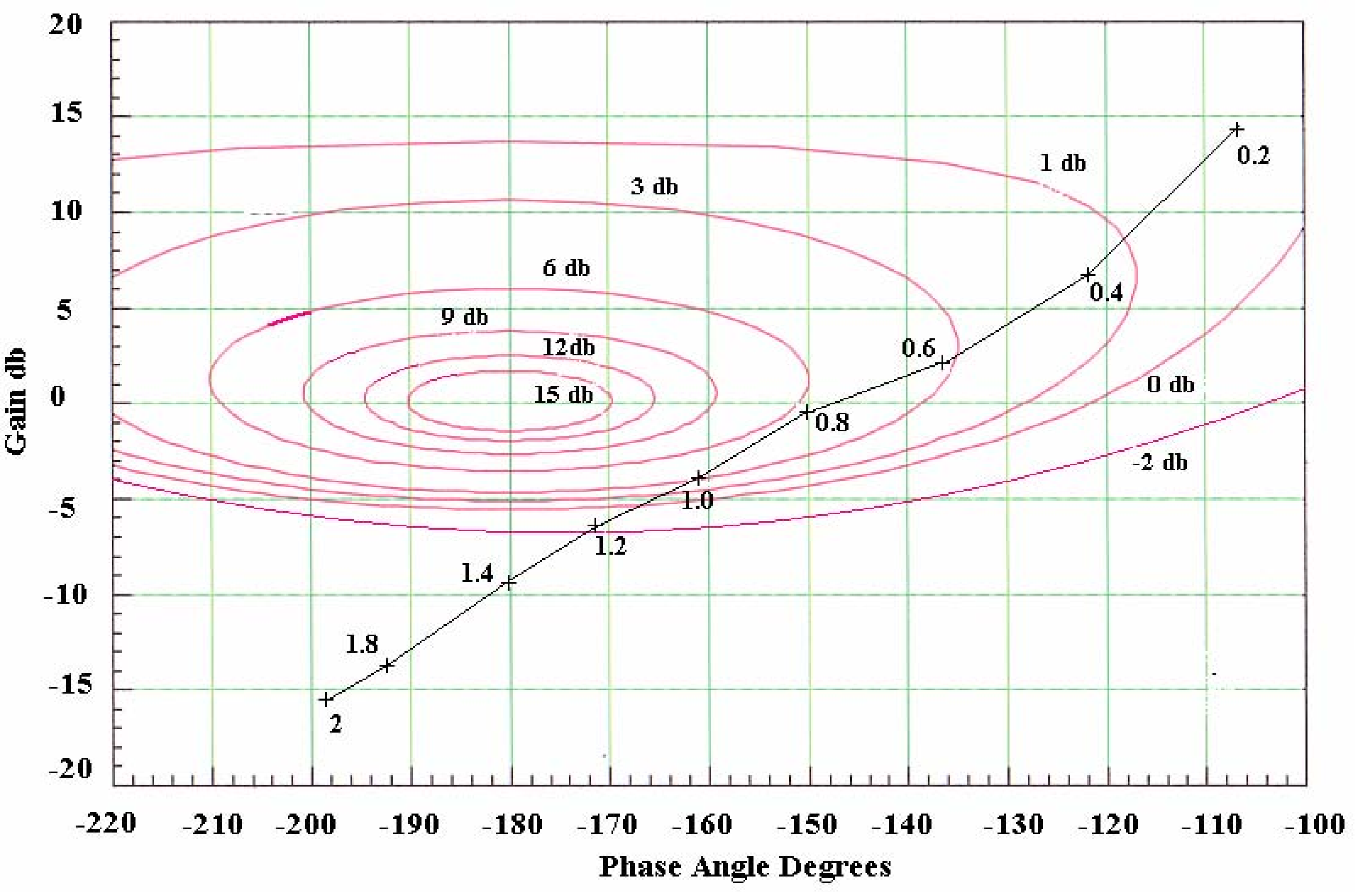
Evaluate the gain and phase angles for $G_1 = \dfrac{2}{s}$ $G_2 = \dfrac{1}{(s+1)}$ and $G_3 = \dfrac{1}{(s+2)}$

A suitable frequency range is 0.4 to 4 rad/s.

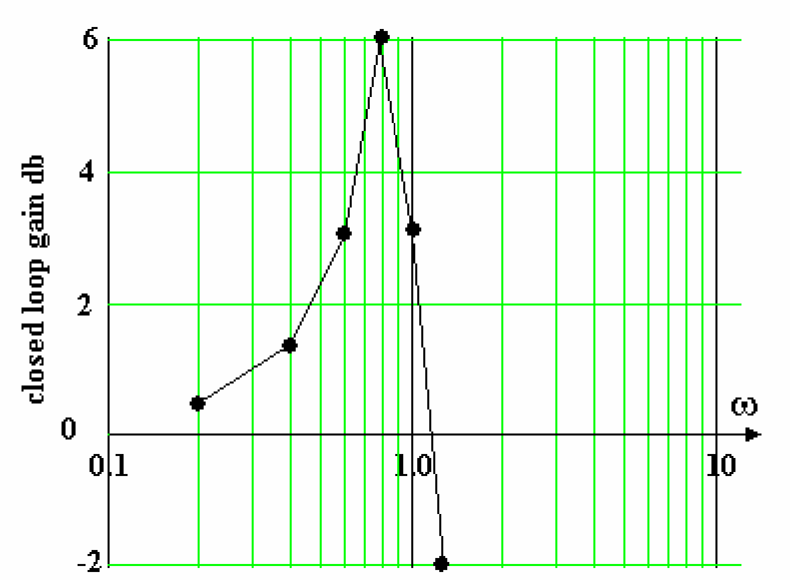
In general if we have $\dfrac{K}{n + Ts}$ the resulting gain and angle are $\dfrac{K}{\sqrt{n^2 + \omega^2 T^2}}$ and $-\tan^{-1}\left(\dfrac{\omega T}{n}\right)$

Remember that gains in db are added and angles are the product. In this case it might be as easy to use $G\,db = 20\log(G_1 G_2 G_3)$

ω	G_1	G_2	G_3	G db	ϕ_1	ϕ_2	ϕ_3	ϕ
0.2	10	0.498	0.981	13.766	-89.971	-5.711	-11.31	-106.992
0.4	5	0.49	0.928	7.144	-89.986	-11.31	-21.801	-123.097
0.6	3.333	0.479	0.857	2.727	-89.99	-16.699	-30.964	-137.653
0.8	2.5	0.464	0.781	-0.855	-89.993	-21.801	-38.66	-150.454
1	2	0.447	0.707	-3.979	-89.994	-26.565	-45	-161.559
1.2	1.667	0.429	0.64	-6.793	-89.995	-30.964	-50.194	-171.153
1.4	1.429	0.41	0.581	-9.367	-89.996	-34.992	-54.462	-179.45
1.6	1.25	0.39	0.53	-11.745	-89.996	-38.66	-57.995	-186.651
1.8	1.111	0.372	0.486	-13.956	-89.997	-41.987	-60.945	-192.929
2	1	0.354	0.447	-16.021	-89.997	-45	-63.435	-198.432



Now pick off the closed loop gain at various frequencies and plot the frequency response. It is typical of a second order system.



The peak magnification is at 0.8 rad/s and the closed loop gain is 6 db.

From the rubric of the exam paper we are given $M_p(\omega) = 2\xi\sqrt{(1-\xi^2)}^{-1}$ and $\omega_r = \omega_n\sqrt{(1-\xi^2)}$

$20 \log M_p = 6 \quad M_p = 2$

$$2 = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$4\xi\sqrt{1-\xi^2} = 1$$

$$16\xi^2\left(1-\xi^2\right) = 1$$

Let $= \xi^2 = x$ and solve x

$16x\,(1 - x) = 1 \qquad -16x^2 + 16x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-16 \pm \sqrt{16^2 - 4x16x1}}{-32} = \frac{-16 \pm 13.86}{-32} = 0.933 \text{ or } 0.067$$

$\xi = \sqrt{x} = 0.966$ or 0.259

Using 0.259 we get

$\omega_r = \omega_n\sqrt{(1-\xi^2)} \quad \omega_n = 0.8/\sqrt{(1- 0.259^2)} = 0.828$ rad/s

The standard second order transfer function is $G(s) = \dfrac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2}$ and putting In the values found we get $G(s) = \dfrac{0.69}{s^2 + 0.43s + 0.69}$. The examiner gives $G(s) = \dfrac{0.77}{s^2 + 0.5s + 0.77}$

(c) SOLUTION

The settling time is found from the equation $e^{\zeta\omega_n t_s} = \%\text{ overshoot} = 0.02$ (Not given in the paper).

$\ln(0.02) = -3.92 = \xi\,\omega_n\,t_s$

$t_s = 3.912/(0.259 \times 0.8) = 18.9$ s (The examiner gives 15.75 secs)

In retrospect the peak was about 0.83 rad/s which would produce a better answer.

ALTERNATIVE SOLUTION

(a) **SOLUTION**

A Nyquist Plot would do this but let's use the Routh Hurwitz criterion. First multiply out.

$G(s) = 2/(s^3 + 3s^2 + 2s)$

The closed loop transfer function for unity feed back is $G_{cl} = 1/\{G(s) + 1\}$

Characteristic equation is $\frac{2}{s^3 + 3s^2 + 2s} + 1 = 0$ hence $s^3 + 3s^2 + 2s + 2 = 0$

If the highest power of the characteristic equation is 3 the criteria may be simplified as follows.

Characteristic equation = $(as^3 + bs^2 + cs + d)$ so a = 1 b = 3 c = 2 d = 2

If the next coefficient is negative the system is unstable and this is given by R = c - ad/b

R= 2- (1 x 2/3) = 2 – 2/3 = 1.33 This is positive so the system is stable.

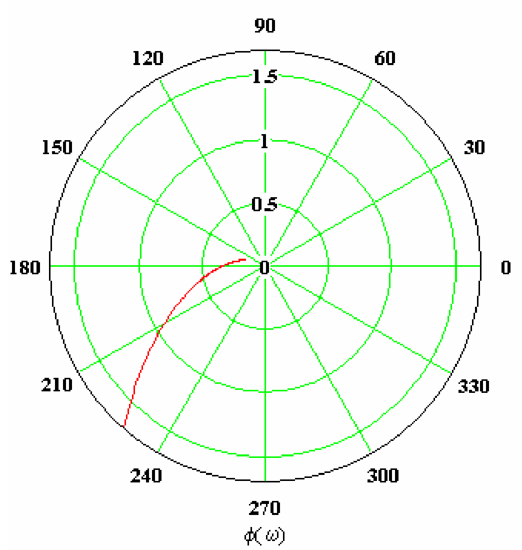
(b) **SOLUTION**

Evaluate the gain and phase angles for $G_1 = \frac{2}{s}$ $G_2 = \frac{1}{(s + 1)}$ and $G_3 = \frac{1}{(s + 2)}$

A suitable frequency range is 0.1 to 1.6 rad/s. In general if we have $\frac{K}{n + Ts}$ the resulting gain and angle are $\frac{K}{\sqrt{n^2 + \omega^2 T^2}}$ and $-\tan^{-1}\left(\frac{\omega T}{n}\right)$ To find the overall result we use $G = G_1\ G_2\ G_3$ and $\phi = \phi_1 + \phi_2 + \phi_3$ A suitable frequency range must be found by experimenting with figures. Here is the result.

ω	G_1	G_2	G_3	G	ϕ_1	ϕ_2	ϕ_3	ϕ
0.1	19.999	0.499	0.995	9.937	-89.427	-2.862	-5.711	-98
0.2	10	0.498	0.981	4.879	-89.714	-5.711	-11.31	-106.734
0.3	6.667	0.494	0.958	3.157	-89.809	-8.531	-16.699	-115.039
0.4	5	0.49	0.928	2.276	-89.857	-11.31	-21.801	-122.968
0.5	4	0.485	0.894	1.735	-89.885	-14.036	-26.565	-130.487
0.6	3.333	0.479	0.857	1.369	-89.905	-16.699	-30.964	-137.568
0.7	2.857	0.472	0.819	1.105	-89.918	-19.29	-34.992	-144.2
0.8	2.5	0.464	0.781	0.906	-89.928	-21.801	-38.66	-150.39
0.9	2.222	0.456	0.743	0.753	-89.936	-24.228	-41.987	-156.151
1	2	0.447	0.707	0.632	-89.943	-26.565	-45	-161.508
1.1	1.818	0.438	0.673	0.536	-89.948	-28.811	-47.726	-166.485
1.2	1.667	0.429	0.64	0.457	-89.952	-30.964	-50.194	-171.11
1.3	1.538	0.419	0.61	0.393	-89.956	-33.024	-52.431	-175.411
1.4	1.429	0.41	0.581	0.34	-89.959	-34.992	-54.462	-179.413
1.5	1.333	0.4	0.555	0.296	-89.962	-36.87	-56.31	-183.142
1.6	1.25	0.39	0.53	0.259	-89.964	-38.66	-57.995	-186.619

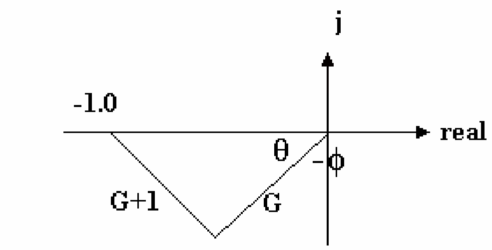
The Nyquist plot verifies that the system is stable since the -1 point is not enclosed. (This needs more results around ω = 0.5)



We must calculate the magnification factor M for the closed loop system. With unitary feedback we know

$$G_{cl} = \frac{G(s)}{1+G(s)} \quad \text{and } M = \frac{|G|}{|1+G|}$$

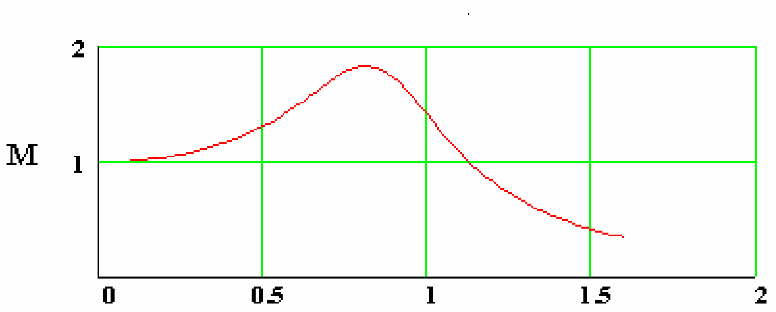
The value of G+1 is found from the vector for G as follows



$$M = \frac{G}{G+1} \qquad G+1 = \sqrt{\{G\sin\theta\}^2 + \{G(1-\cos\theta)\}^2}$$

This enables us to calculate M and the results are as follows.

ω	ϕ	G	θ	G+1	M
0.1	-98	9.937	82	9.848	1.009
0.2	-106.734	4.879	73.266	4.689	1.04
0.3	-115.039	3.157	64.961	2.88	1.096
0.4	-122.968	2.276	57.032	1.924	1.183
0.5	-130.487	1.735	49.513	1.326	1.309
0.6	-137.568	1.369	42.432	0.924	1.482
0.7	-144.2	1.105	35.8	0.654	1.688
0.8	-150.39	0.906	29.61	0.495	1.829
0.9	-156.151	0.753	23.849	0.435	1.73
1	-161.508	0.632	18.492	0.448	1.413
1.1	-166.485	0.536	13.515	0.495	1.082
1.2	-171.11	0.457	8.89	0.553	0.828
1.3	-175.411	0.393	4.589	0.609	0.646
1.4	-179.413	0.34	0.587	0.66	0.515
1.5	-183.142	0.296	-3.142	0.705	0.42
1.6	-186.619	0.259	-6.619	0.744	0.348



The result resembles the response of a second order system where the peak M_p = 1.83 and the resonant frequency is ω_r = 0.8 rad/s

From the rubric of the exam paper we are given $M_p(\omega) = 2\xi\sqrt{(1-\xi^2)}^{-1}$ and $\omega_r = \omega_n\sqrt{(1-\xi^2)}$

$$1.83 = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$3.66\xi\sqrt{1-\xi^2} = 1$$

$$13.4\xi^2\left(1-\xi^2\right) = 1$$

Let $= \xi^2 = x$ and solve x

13.4x (1 – x) = 1 $\quad$ $-13.4\,x^2 + 13.4\,x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-13.4 \pm \sqrt{13.4^2 - 4\,x\,13.4\,x\,1}}{-26.8} = \frac{-13.4 \pm 11.22}{-26.8} = 0.0814 \text{ or } 0.919$$

$\xi = \sqrt{x}$ = 0.285 or 0.958 The higher figure does not look right as there would be very little peak.

Using 0.2853 we get

$\omega_r = 0.8 = \omega_n\sqrt{(1-\xi^2)}$ $\quad$ $\omega_n = 0.8/\sqrt{(1- 0.285^2)} = 0.835$ rad/s

The standard second order transfer function is $G(s) = \dfrac{\omega_n^2}{s^2 + 2s\,\zeta\,\omega_n + \omega_n^2}$ and putting In the values found we get $G(s) = \dfrac{0.7}{s^2 + 0.43s + 0.7}$. The examiner gives $G(s) = \dfrac{0.77}{s^2 + 0.5s + 0.77}$

(c) The settling time is found from the equation $e^{\zeta\,\omega_n\,t_s} = \%\text{ overshoot} = 0.02$ (Not given in the paper).

ln (0.02) = -3.92 = $\xi\,\omega_n\,t_s$

t_s = 3.912/(0.285 x 0.835) = 16.4 s (The examiner gives 15.75 secs)