Q4 2008

a A circular pipe has a vertical axis. Oil spills over the open top of the pipe at a steady rate and flows down the outside of the pipe under gravity, forming a symmetrical and continuous film. A short distance down the outside of the pipe from the open top the film becomes fully developed with a constant film thickness. By choosing an axi-symmetric element of fluid in the fully developed region of the oil film, show that the following equation applies for laminar flow in the film

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\frac{\rho g r}{\mu}$$

where v is the fluid velocity at radius r in the film, ρ is the density and μ the dynamic viscosity of the fluid and g the gravitational acceleration.

b. Using the result in part a above and assuming negligible drag on the oil by the surrounding air, show that the fluid velocity v at radius r in the fully developed film s given by

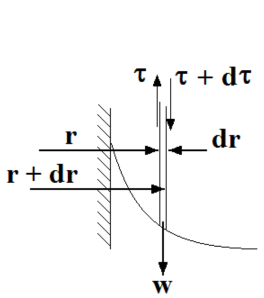
$$\frac{\rho g r_f^2}{4\mu}\left[2ln\left(\frac{r}{r_p}\right) - \left(\frac{r^2 - r_p^2}{r_f^2}\right)\right]$$

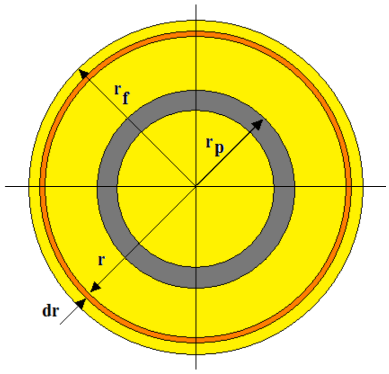
where r_f and r_p are the radii at the film surface and the pipe outside surface respectively.

c. Calculate the volume flow rate of oil required to maintain a film thickness of 5mm when the outside diameter of the pipe is 100 mm, given that μ = 0.052 N s/m and ρ = 870 kg/m^3 for the oil.

Note: $\int x ln x dx = \frac{1}{4}x^2(2ln x - 1)$

SOLUTION





The key to this is recognising that the weight of the oil has to overcome the viscous drag. In an exam, don't spend too much time on part 1 if you can't get the correct answer. Go on to part 2 and 3 which you can do using the information provided in the question.

Consider an elementary thin cylindrical layer that makes an element of flowing down the outside of the pipe. The length is δx , the inside radius is r and the radial thickness is dr. The weight of the element is w and the shear stress on the surface increases by dτ from the inner to the outer surface. The velocity at any point is v and the dynamic viscosity is μ.

The weight is ρ g x volume $\quad w = \rho g \delta x \{\pi(r+dr)^2 - \pi r^2\}$

The shear force opposing is $\quad F = \{(\tau+\delta\tau)(2\pi)(r+dr) - \tau 2\pi r\}\delta x$

Force balance gives $\quad \rho g \{\pi(r + dr)^2 - \pi r^2\} + \{(\tau+\delta\tau)(2\pi)(r+dr) - \tau 2\pi r\}\delta x = 0$

$$\rho g \delta x \pi\{(r + dr)^2 - r^2\} + 2\pi\{(\tau+d\tau)(r+dr) - \tau r\}\delta x = 0$$

$$\rho g \{(r + dr)^2 - r^2\} + 2\{(\tau+d\tau)(r+dr) - \tau r\} = 0$$

Multiply out $\quad \rho g\{r^2 + (dr)^2 + 2rdr - r^2\} + 2(\tau r + \tau dr + d\tau\, r + d\tau dr - \tau r) = 0$

Ignore small products $\quad \rho g\{rdr\} + (\tau dr + d\tau\, r) = 0$

$$-\rho g r = \tau + r\frac{d\tau}{dr}$$

Substitute $\quad \tau = \mu\frac{dv}{dr}$

$$-\rho g r = \mu\frac{dv}{dr} + \mu r\frac{d\left(\frac{dv}{dr}\right)}{dr}$$

$$-\frac{\rho g r}{\mu} = \frac{dv}{dr} + r\frac{d\left(\frac{dv}{dr}\right)}{dr}$$

Partial differentiation shows that $\quad \frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{dv}{dr} + r\frac{d_2 v}{dr^2} = \frac{dv}{dr} + r\frac{d\left(\frac{dv}{dr}\right)}{dr}$

Hence $\quad \boldsymbol{-\frac{\rho g r}{\mu} = \frac{d}{dr}\left(r\frac{dv}{dr}\right)}$

b. $$-\frac{\rho g r}{\mu} = \frac{d}{dr}\left(r\frac{dv}{dr}\right)$$

Integrate $$-\frac{\rho g r^2}{2\mu} + A = r\frac{dv}{dr}$$

The gradient is zero when r = r_f $$-\frac{\rho g r_f^2}{2\mu} + A = 0 \qquad \frac{\rho g r_f^2}{2\mu} = A$$

$$-\frac{\rho g r^2}{2\mu} + A = r\frac{dv}{dr} \qquad -\frac{\rho g r}{2\mu} + \frac{A}{r} = \frac{dv}{dr}$$

Integrate

$$-\frac{\rho g r^2}{4\mu} + A ln r + B = v$$

$$-\frac{\rho g r^2}{4\mu} + \frac{\rho g r_f^2}{2\mu} ln r + B = v$$

v = 0 when r = r_p

$$-\frac{\rho g r_p{}^2}{4\mu} + \frac{\rho g r_f^2}{2\mu} ln r_p + B = 0$$

$$\frac{\rho g r_p{}^2}{4\mu} - \frac{\rho g r_f^2}{2\mu} ln r_p = B \qquad \frac{\rho g}{4\mu}\left(r_p{}^2 - 2r_f^2 ln r_p\right) = B$$

$$-\frac{\rho g r^2}{4\mu} + \frac{\rho g r_f^2}{2\mu} ln r + \frac{\rho g}{4\mu}\left(r_p{}^2 - 2r_f^2 ln r_p\right) = v$$

$$\frac{\rho g}{4\mu}\left(-r^2 + 2r_f^2 ln r\right) + \left(r_p{}^2 - 2r_f^2 ln r_p\right) = v$$

$$\frac{\rho g}{4\mu}\left(r_p{}^2 - r^2\right) + 2r_f^2\left(ln r - ln r_p\right) = v$$

$$\boldsymbol{\frac{\rho g r_f^2}{4\mu}\left\{2\left(ln\frac{r}{r_p}\right) - \left(\frac{r^2 - r_p{}^2}{r_f^2}\right)\right\} = v}$$

c. Volume flow rate through the elementary ring is v (2πr dr)

Total flow is $$Q = \frac{\pi\rho g r_f^2}{2\mu}\int_{r_f}^{r_p}\left\{2r\left(ln\frac{r}{r_p}\right) - r\left(\frac{r^2 - r_p{}^2}{r_f^2}\right)\right\}dr$$

$$Q = \frac{\pi\rho g r_f^2}{2\mu}\int_{r_f}^{r_p}\left\{2\left(r ln\frac{r}{r_p}\right) - \frac{r^3}{r_f^2} + \frac{r\, r_p{}^2}{r_f^2}\right\}dr$$

$$Q = \frac{\pi\rho g r_f^2}{2\mu}\left[\left(r^2 ln\left(\frac{r}{r_p}\right) - \frac{r^2}{2}\right) - \frac{r^4}{4r_f^2} + \frac{r^2\, r_p{}^2}{2r_f^2}\right]_{r_f}^{r_p}$$

$$Q = \frac{\pi(870)(9.81)(0.055)^2}{2(0.052)}\left[\left(r^2 ln\left(\frac{r}{r_p}\right) - \frac{r^2}{2}\right) - \frac{r^4}{4r_f^2} + \frac{r^2\, r_p{}^2}{2r_f^2}\right]_{r_f}^{r_p}$$

$$Q = 878.48\left[r^2 ln\left(\frac{r}{r_p}\right) - \frac{r^2}{2} - \frac{r^4}{4r_f^2} + \frac{r^2\, r_p{}^2}{2r_f^2}\right]_{0.055}^{0.05}$$

Evaluate and Q = 2.667 x 10^{-3} m^3/s