APPLIED FLUID MECHANICS D203 QUESTION 6 2004-08-18

(a) For isentropic flow in a variable area duct, derive the expression

$$\frac{dA}{A} = \frac{du}{u}\left(M^2 - 1\right)$$

where u is the fluid velocity, M the Mach number and A the cross-sectional area of the duct. Use the expression to explain why a convergent/divergent nozzle is required to produce a supersonic flow from a reservoir of stationary gas.

(b) Air in a reservoir has a pressure of 500 kN/m^2 and temperature of 20 °C. It is connected to a receiver by a convergent nozzle with exit diameter 35 mm. If the pressure in the receiver is maintained at 300 kN/m^2, calculate

(i) the temperature and density of the air at exit from the nozzle
(ii) the exit Mach number
(iii) the air mass flow rate through the nozzle.

(c) Repeat the calculations in part (b) when the receiver pressure is 200 kN/m^2. The following equations may be used where appropriate without proof.

$$T_o = T\left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right] \qquad \frac{p}{p_o} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma - 1}} \qquad a = \sqrt{\gamma RT} \qquad m = \rho A u$$

SOLUTION

w = specific work q = specific heat transfer u = velocity v = specific volume h = specific enthalpy
p = pressure s = specific entropy T = temperature a = sonic velocity M = u/a

CONSERVATION OF ENERGY

dq + dw = dh + d(ke) but since the flow is isentropic q = 0 and since no work is done w = 0

$dh + d(u^2/2) = 0$ dh = T ds + v dp but since it is isentropic ds = 0

dh + u du= 0 dh = v dp

v dp + u du = 0

v dp = - u du ……………(A)

$du/u = - v\,dp/u^2$…………(B)

CONSERVATION OF MASS

A u/v = constant take logs

log A + log u - log v = const differentiate

dA/A + du/u – dv/v = 0

dA/A = dv/v – du/u substitute (A)

$dA/A = dv/v + v\,dp/u^2$

$$\frac{dA}{A} = v dp\left(\frac{dv}{v^2 dp} + \frac{1}{u^2}\right) \quad \text{It can be shown that } a^2 = -v^2\frac{dp}{dv}$$

$$\frac{dA}{A} = v dp\left(-\frac{1}{a^2} + \frac{1}{u^2}\right) = \frac{v dp}{u^2}\left(-\frac{u^2}{a^2} + 1\right)$$

$$\frac{dA}{A} = \frac{v dp}{u^2}\left(1 - M^2\right) \quad \text{substitute (B)} \quad \frac{dA}{A} = \frac{du}{u}\left(M^2 - 1\right)$$

If M 〉 1 the flow is supersonic so dA/A must be positive – i.e. divergent.
If M 〈 1 the flow is subsonic so dA/A must be negative – i.e. convergent.

(b) $T_o = 293$ K $p_o = 500$ kPa $p_e = 300$ kPa $A_e = \pi \times 0.035^2/4 = 962.1 \times 10^{-6}$ m^2

$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{500}{300} = \left(\frac{293}{T_e}\right)^{\frac{1.4}{1.4-1}} = \left(\frac{293}{T_e}\right)^{3.5} \qquad 1.157 = \frac{293}{T_e} \quad T_e = 253.2\text{K}$$

$$\frac{T_o}{T} = 1 + \left(\frac{\gamma-1}{2}\right)M_e^2 = \frac{293}{253.2} = 1.157$$

$$\left(\frac{\gamma-1}{2}\right)M_e^2 = 0.157 = 0.2M_e^2$$

$$M_e^2 = 0.786 \quad M_e = 0.886$$

$$a_e = \sqrt{\gamma R T_e} = \sqrt{1.4 \text{ x } 287 \text{ x } 253.2} = 319 \text{ m/s}$$

$$u_e = 0.886 \text{ x } 319 = 282.7\text{m/s}$$

$$\rho_e = \frac{p_e}{RT_e} = \frac{300\,000}{287 \text{ x } 253.2} = 4.128 \text{ kg/m}^3$$

m = ρ A u = 4.128 x 962.1 x 10^{-6} x 282.7 =1.123 kg/s

(c) $p_e = 200$ kPa The pressure ratio is 200/500 = 0.4

$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{500}{200} = \left(\frac{293}{T_e}\right)^{\frac{1.4}{1.4-1}} = \left(\frac{293}{T_e}\right)^{3.5} \qquad 1.3 = \frac{293}{T_e} \quad T_e = 225.3\text{K}$$

$$\frac{T_o}{T_e} = 1 + \left(\frac{\gamma-1}{2}\right)M_e^2 = \frac{293}{225.3} = 1.3$$

$$\left(\frac{\gamma-1}{2}\right)M_e^2 = 0.3 = 0.2M_e^2$$

$$M_e^2 = 0.786 \quad M_e = 1.225$$

The velocity is supersonic so the nozzle is choked so calculating the mass flow in the same way as part (b) is invalid.

The critical pressure ratio is $\frac{p_o}{p_t} = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528$

When choked the pressure at the throat is the critical value and Mach number is 1.0

$$\frac{p_o}{p_t} = \left(\frac{\gamma}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528 \quad p_t = 0.528 \text{ x } 500 \text{ kPa} = 264.1 \text{ kPa}$$

$$\frac{T_o}{T_t} = 1 + \left(\frac{\gamma-1}{2}\right)M_e^2 = 1.2 \quad T_t = 244.2 \text{ K}$$

$$\rho_t = \frac{p_t}{RT_t} = \frac{264\,100}{287 \text{ x } 244.2} = 3.769 \text{ kg/m}^3$$

We need the throat area. This is found from

$$A_t = A_e M_e \left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}} = 962.1\text{x}10^{-6} \text{ x } 1.225\left(\frac{200}{264.1}\right)^{0.857} = 928.6 \text{ x } 10^{-6}\text{m}^2$$

$$a_t = \sqrt{\gamma R T_t} = \sqrt{1.4 \text{ x } 287 \text{ x } 244.2} = 313.2 \text{ m/s}$$

$$u_e = 313.2 \text{ m/s}$$

m = ρ A u = 3.769 x 928.6 x 10^{-6} x 313.2 =1.1 kg/s This should be more than in part (b) so an error somewhere???