APPLIED MECHANICS OF FLUIDS D203 Q4 2004

a) Show that the combination of a uniform flow with velocity U in the x-direction, a doublet of strength Q at the origin of an x - y coordinate system and an irrotational vortex with circulation Γ also at the origin can be used to describe the inviscid flow of a fluid around a rotating cylinder. Derive expressions for the diameter of the cylinder and its rotational speed.

b) For the flow conditions corresponding to the combination given in part (a), show that
(i) the drag exerted on the cylinder by the flow is zero
(ii) the lift experienced by the cylinder is $\rho U \Gamma$ per unit length of cylinder, where ρ is the fluid density.

c) A long 200 mm diameter cylinder is rotating at 1000 rev/min about a vertical axis in a steady stream of air which is flowing horizontally with a velocity of 5 m/s. The pressure and temperature of the air far upstream of the cylinder are 1.0 bar and 20°C respectively. Calculate
(i) the position of any stagnation points in the flow
(ii) the lift force on the cylinder per unit length and its direction
(iii) the minimum pressure and its location on the surface of the cylinder.

Note $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$ and $\int \sin^3 x = -\frac{1}{3}\left(2+\sin^2 x\right)\cos x$

DOING THIS QUESTION IN THE TIME ALLOTTED REQUIRES A GENIUS

(a) The stream function for UNIFORM FLOW + DOUBLET is

$\Psi = -Uy + (Q/\pi r)\sin\theta$ (depending on notation used)
$y = r\sin\theta \quad Q/\pi = B$
Q is the flow per unit depth of the source and sink making the doublet.
$\Psi = -Ur\sin\theta + B\sin\theta/r = -\sin\theta(Ur - B/r)$
In order that the cylinder be a solid surface $\Psi = 0$ hence $U r = B/r \quad B = U r^2$
And let this radius be R_o so $B = U R_o^2$

The diameter of the cylinder is hence $D = 2R_o = 2\sqrt{(B/U)} = 2\sqrt{(Q/\pi U)}$

$\Psi = -\sin\theta(Ur - U R_o^2/r^2 = -U\sin\theta\,(r - R_o^2/r)$

If we add a vortex at the origin, a circular motion is added giving the effect of a rotating cylinder. The velocity of the vortex at the same radius as the cylinder must be the tangential velocity of the cylinder.
For a vortex $ur = \omega r^2 = C = \omega R_o^2$ at the cylinder surface. u is the velocity of the stream line.
Γ = circulation $= 2\pi\omega R_o^2 = 2\pi C$
$\omega = \Gamma/2\pi R_o^2$ $\quad N = \omega/2\pi = \Gamma/4\pi^2 R_o^2$ $\quad R_o = \sqrt{(\Gamma/4N\pi^2)}$ $\quad \Gamma = 4N\pi^2 R_o^2$

For a free vortex $\Psi = C\ln(r/a) = u\,r\ln(r/a)$ a is the inner radius of the vortex.

The total stream function is hence $\Psi = -U\sin\theta\,(r - R_o^2/r) + C\ln(r/a)$

The tangential velocity is $v_T = d\Psi/dr = -U\sin\theta\left\{1+\frac{R_o^2}{r^2}\right\}+\frac{C}{r}$

At the cylinder surface $r = R_o$ so $v_T = -2U\sin\theta + \frac{C}{R_o}$

The stagnation points occur where this is zero so $\sin\theta = C/2UR_o$

PRESSURE DISTRIBUTION

Apply Bernoulli between a point in the undisturbed flow and a point on the cylinder.

$$p_o - p = \frac{\rho}{2}\left(v_T^2 - U^2\right) = \frac{\rho}{2}\left\{\left(-U\sin\theta\left\{1+\frac{R_o^2}{r^2}\right\}+\frac{C}{r}\right)^2 - U^2\right\}$$

$$p_o - p = \frac{\rho}{2}\left(4U^2\sin^2\theta - \frac{4CU\sin\theta}{R_o} + \frac{C^2}{R_o^2} - U^2\right)$$

$$p_o - p = \frac{\rho U^2}{2}\left(4\sin^2\theta - \frac{4C\sin\theta}{UR_o} + \frac{C^2}{UR_o^2} - 1\right) \text{ and simplifying by putting } \frac{C^2}{U^2R_o^2} = \beta^2$$

$$p_o - p = \frac{\rho U^2}{2}\left(4\sin^2\theta - 4\,\beta\sin\theta + \beta^2 - 1\right)$$

DRAG integrate the pressure force acting on the surface horizontally.

$$D = \int_0^{2\pi}(p_o - p)R_o\cos\theta\,d\theta = \int_0^{2\pi}\frac{\rho U^2}{2}\left(4\sin^2\theta - 4\,\beta\sin\theta + \beta^2 - 1\right)R_o\cos\theta\,d\theta$$

$$D = \frac{R_o\rho U^2}{2}\int_0^{2\pi}\left(4\sin^2\theta\cos\theta - 4\,\beta\sin\theta\cos\theta + \beta^2\cos\theta - \cos\theta\right)d\theta$$

There may be a short cut but take it on advice that this comes to zero. Note that the drag in an inviscid fluid should always be zero.

LIFT integrate the pressure force acting on the surface vertically.

$$L = \int_0^{2\pi}(p_o - p)R_o\sin\theta\,d\theta = \int_0^{2\pi}\frac{\rho U^2}{2}\left(4\sin^2\theta - 4\,\beta\sin\theta + \beta^2 - 1\right)R_o\sin\theta\,d\theta$$

$$L = \frac{R_o\rho U^2}{2}\int_0^{2\pi}\left(4\sin^3\theta - 4\,\beta\sin^2\theta + \beta^2\sin\theta - \sin\theta\right)d\theta$$

Using the identities given in the question we obtain

$$L = \frac{R_o\rho U^2}{2}\left[\frac{-4}{3}\left(2+\sin^2\theta\right)\cos\theta - 4\beta\left(\frac{\theta}{2} - \frac{\sin(2\theta i}{4}\right) - \beta^2\cos\theta - \sin\theta\right]_0^{2\pi}$$

Integrating each part separately between 0 and 2π

$$\left[\frac{-4}{3}\left(2+\sin^2\theta\right)\cos\theta\right]_0^{2\pi} = \left[\frac{-4}{3}(2+0)\right] - \left[\frac{-4}{3}(2+0)\right] = 0$$

$$\left[-4\beta\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right)\right]_0^{2\pi} = \left[-4\beta\left(\frac{2\pi}{2} - \frac{\sin(4\theta i}{4}\right)\right] - \left[-4\beta\left(\frac{0}{2} - \frac{\sin(0)}{4}\right)\right]$$

$$= \left[-4\beta(\pi - 0)\right] - [0] = -4\beta\,\pi$$

$$\left[-\beta^2\cos\theta\right]_0^{2\pi} = \left[-\beta^2\cos(2\pi)\right] - \left[-\beta^2\cos(0)\right] = 0$$

$$\left[-\sin\theta\right]_0^{2\pi} = 0$$

$$L = \frac{R_o\rho U^2}{2}\left[0 - 4\,\beta\,\pi - 0 - 0\right] = R_o\rho U^2\,\beta\,2\pi = R_o\rho U^2\;2\pi\frac{C}{UR_o} = \rho U\;2\pi\,C$$

$2\pi C$ is the circulation Γ so $L = \rho U\Gamma$

c. Dia = 200 mm $R_o = 0.1$ m N = 1000/60 rev/s $\omega = 2\pi N = 104.72$ rad/s U = 5 m/s.

$\Gamma = 2\pi\omega R_o^2 = 2\pi \times 104.72 \times 0.1^2 = 6.58$

$C = \Gamma/2\pi = 1.047$

$\sin\theta = \frac{C}{2UR_o} = \frac{1.047}{2 \times 5 \times 0.1} = 1.047$ as the maximum value can only be 1.0 there is an error or perhaps the stagnation point is off the surface.

p = 1.0 bar T = 293 K $\rho = \frac{p}{RT} = \frac{1 \times 10^5}{287 \times 293} = 1.189 \text{ kg/m}^3$

$L = \rho U \Gamma = 1.189 \times 5 \times 1.047 = 6.22$ N/m

$p = p_o - \frac{\rho}{2}\left(v_T^2 - U^2\right)$ and this will be a minimum when v_T is a maximum

$$v_T = -U\sin\theta\left\{1 + \frac{R_o^2}{r^2}\right\} + \frac{C}{r} = -2 \times 5 \times \sin\theta + \frac{1.047}{0.1} = 10.47 - 10\sin\theta$$

This will be a maximum when $\sin\theta = 0$ and $v_T = 10.47$ m/s

$p = p_o - \frac{\rho}{2}\left(v_T^2 - U^2\right) = 1 \times 10^5 - \frac{1.189}{2}\left(10.47^2 - 10^2\right)$ the change in pressure is tiny

I suspect an error somewhere but can't find it. Anyone able to help please contact me.