APPLIED FLUID MECHANICS D203 SOLUTIONS 2004 – QUESTION 2

(a) A viscous liquid with density 800 kg/m^3 and dynamic viscosity 0.25 Ns/m^2 flows upwards in the annular space between two concentric vertical pipes with diameters 300 mm and 200 mm respectively.

(i) Assuming that the flow is fully developed and laminar, show that the flow rate Q is given by

$$Q = \frac{-\pi}{8\mu}\left(\frac{dp}{dz} + \rho g\right)\left[\left\{R_o^4 - R_i^4\right\} - \frac{\left(R_o^2 - R_i^2\right)^2}{\ln\left\{\frac{R_o}{R_i}\right\}}\right]$$

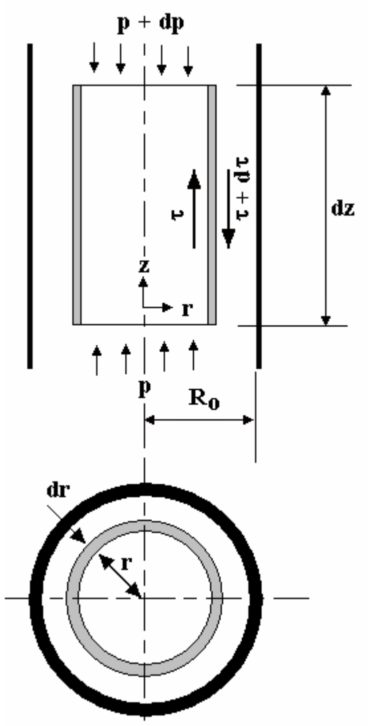
where dp/dz is the pressure gradient in the flow direction, p and μ are the liquid density and dynamic viscosity respectively, g is gravitational acceleration and R_i and R_o are the inner and outer radii of the annular space.

(ii) Calculate the pressure drop over one metre length of the vertical pipes when the flow rate is 0.050 m^3/s.

(iii) Verify that the flow is laminar for these conditions.

(b) For the flow described in part (a) above, calculate the maximum velocity of the liquid in the annular space and the radius at which it occurs.

COMMENT It is unrealistic to do the complete derivation in a reasonable amount of time unless there is a short cut method unknown to me. It may be acceptable to start with equation B which should be remembered and can be applied to a range of circumstances.



Consider fluid flowing vertically in a pipe. Consider a stream tube of length dz at radius r and thickness dr. The pressure vertically decreases as ρgz in the z direction.

The cross sectional area of the thin circular ring is 2πr dr
The surface area of the elementary cylinder is 2πr dz at the inside and 2π(r+dr) dz on the outside.

The shear force is 2π(r+dr) dz (τ + dτ) - 2π(r) dz (τ) which simplifies down to 2π dz (τ dr + r dτ)

Balancing forces we have

$$2\pi r\, dr\, p = 2\pi r\, dr\, (p + dp) + 2\pi r\, dr\, \rho g dz + 2\pi\, dz\, (\tau\, dr + r\, d\tau)$$

$$r\, dr\, p = pr\, dr + dp\, r\, dr + r\, dr\, \rho g dz + dz\, (\tau\, dr + r\, d\tau)$$

$$- dp\, r\, dr - r\, dr\, \rho g dz = dz\, (\tau\, dr + r\, d\tau)$$

$$-\frac{dp}{dz} - \rho g = \frac{\tau}{r} + \frac{d\tau}{dr}$$

$$-\left(\frac{dp}{dz}+\rho g\right)=\frac{\tau}{r}+\frac{d\tau}{dr} \qquad \tau=-\mu\frac{du}{dy}$$ for Newtonian fluids if the pressure gradient is assumed positive.

If y is measured from the inside of the pipe then r = -y and dy = - dr so $\tau=\mu\frac{du}{dr}$

$$-\left(\frac{dp}{dz}+\rho g\right)=\frac{\mu}{r}\frac{du}{dr}\mu\frac{d^2u}{dr^2}$$

$$\frac{1}{r}\frac{du}{dr}+\frac{d^2u}{dr^2}=-\frac{1}{\mu}\left(\frac{dp}{dz}+\rho g\right)$$

$$\frac{du}{dr}+\frac{rd^2u}{dr^2}=-\frac{r}{\mu}\left(\frac{dp}{dz}+\rho g\right)$$

Using partial differentiation to differentiate $\frac{d\left(r\frac{du}{dr}\right)}{dr}$ yields the result $\frac{du}{dr}+\frac{rd^2u}{dr^2}$

hence $\frac{d\left(r\frac{du}{dr}\right)}{dr}=-\frac{r}{\mu}\left(\frac{dp}{dz}+\rho g\right)$

Integrating we get $r\frac{du}{dr}=-\frac{r^2}{2\mu}\left(\frac{dp}{dz}+\rho g\right)+A$

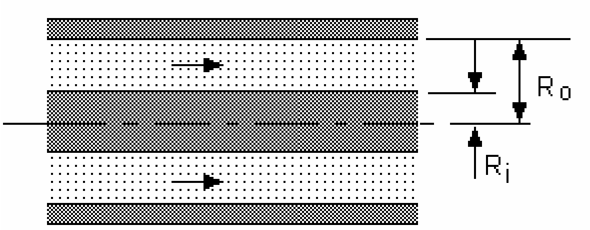
$$\frac{du}{dr}=-\frac{r}{2\mu}\left(\frac{dp}{dz}+\rho g\right)+\frac{A}{r} \quad ....(A)$$

where A is a constant of integration.
Integrating again we get

$$u=-\frac{r^2}{4\mu}\left(\frac{dp}{dz}+\rho g\right)+A\ln r+B................(B)$$

where B is another constant of integration.

Equations (A) and (B) may be used to derive flow through an annular passage.



$$u=-\frac{r^2}{4\mu}\left(\left(\frac{dp}{dz}+\rho g\right)\right)+A\ln r+B$$

The boundary conditions are $u=0$ at $r=R_i$ and $r=R_o$.

$$0=-\frac{R_o^2}{4\mu}\left(\frac{dp}{dz}+\rho g\right)+A\ln R_o+B..........................(C)$$

$$0=-\frac{R_i^2}{4\mu}\left(\frac{dp}{dz}+\rho g\right)+A\ln R_i+B........................(D)$$

Subtract D from C

$$0=\frac{\left(\frac{dp}{dz}+\rho g\right)}{4\mu}\left\{-R_o^2+R_i^2\right\}+A\left\{\ln R_o-\ln R_i\right\}$$

$$0=\frac{\left(\frac{dp}{dz}+\rho g\right)}{4\mu}\left\{R_i^2-R_0^2\right\}+A\ln\left\{\frac{R_o}{R_i}\right\}$$

$$A = \frac{\left(\frac{dp}{dz} + \rho g\right)}{4\mu} \frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}$$

This may be substituted back into equation D. The same result will be obtained from C.

$$0 = -\frac{R_i^2}{4\mu}\left(\frac{dp}{dz} + \rho g\right) + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i + B$$

$$B = \frac{1}{4\mu}\left(\frac{dp}{dz} + \rho g\right)\left[R_i^2 - \left\{\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right] \quad \text{This is put into equation B}$$

$$u = \frac{-r^2}{4\mu}\left(\frac{dp}{dz} + \rho g\right) + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[R_i^2 - \left\{\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right]$$

$$u = \frac{\left(\frac{dp}{dz} + \rho g\right)}{4\mu}\left[-r^2 + \frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + R_i^2 - \frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i\right]$$

$$u = \frac{\left(\frac{dp}{dz} + \rho g\right)}{4\mu}\left[\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln\frac{r}{R_i} + R_i^2 - r^2\right]$$

The flow through the elementary ring is $dQ = 2\pi r\, dr\, u$

$$Q = \frac{2\pi\left(\frac{dp}{dz} + \rho g\right)}{4\mu}\int_{R_i}^{R_o}\left[\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}} r\ln\frac{r}{R_i} + rR_i^2 - r^3\right]$$

$$Q = \frac{\pi\left(\frac{dp}{dz} + \rho g\right)}{2\mu}\left[\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\left(\frac{r^2}{2}\ln\frac{r}{R_i} - \frac{r^2}{4}\right) + \frac{r^2R_i^2}{2} - \frac{r^4}{4}\right]_{R_i}^{R_o}$$

$$Q = \frac{\pi\left(\frac{dp}{dz} + \rho g\right)}{2\mu}\left[\left(\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\left(\frac{R_o^2}{2}\ln\frac{R_o}{R_i} - \frac{R_o^2}{4}\right) + \frac{R_o^2R_i^2}{2} - \frac{R_o^4}{4}\right) - \left(\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\left(\frac{R_i^2}{2}\ln\frac{R_i}{R_i} - \frac{R_i^2}{4}\right) + \frac{R_i^2R_i^2}{2} - \frac{R_i^4}{4}\right)\right]$$

$$Q=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[\left(\frac{\left\{R_o^2-R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\left(2R_o^2\ln\frac{R_o}{R_i}-R_o^2\right)+2R_o^2R_i^2-R_o^4\right)-\left(\frac{\left\{R_o^2-R_i^2\right\}\left(-R_i^2\right)}{\ln\left\{\frac{R_o}{R_i}\right\}}+R_i^4\right)\right]$$

$$Q=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[\left(\left\{R_o^2-R_i^2\right\}\left\{2R_o^2\right\}-\frac{\left\{R_o^2-R_i^2\right\}\left\{R_o^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}+2R_o^2R_i^2-R_o^4\right)-\left(\frac{\left\{-R_o^2R_i^2+R_i^4\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}+R_i^4\right)\right]$$

$$Q=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[\left(\left\{2R_o^4-2R_i^2R_o^2\right\}-\frac{\left\{R_o^2-R_i^2\right\}\left\{R_o^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}+2R_o^2R_i^2-R_o^4\right)+\left(\frac{\left\{R_o^2R_i^2-R_i^4\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}-R_i^4\right)\right]$$

$$Q=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[\left(R_o^4-\frac{\left\{R_o^4-R_i^2R_o^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right)+\left(\frac{\left\{R_o^2R_i^2-R_i^4\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}-R_i^4\right)\right]$$

$$Q=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[R_o^4-R_i^4-\frac{\left\{R_o^4-2R_i^2R_o^2+R_i^4\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right]=\frac{\pi\left(\frac{dp}{dz}+\rho g\right)}{8\mu}\left[R_o^4-R_i^4+\frac{\left(R_o^2-R_i^2\right)^2}{\ln\left\{\frac{R_o}{R_i}\right\}}\right]$$

The result should have a minus in front – anyone able to point out where it has gone please let me know.

(ii) $Q = 0.05\ m^3/s$ $R_o = 0.15\ m$ $R_i = 0.1\ m$ $\mu = 0.25\ Ns/m^2$

$$0.05=\frac{-\pi\left(\frac{dp}{dz}+\rho g\right)}{8\times 0.25}\left[0.15^4-0.1^4+\frac{\left(0.15^2-0.1^2\right)^2}{\ln\left\{\frac{0.15}{0.1}\right\}}\right]$$

$$0.05=\frac{-\pi}{2}\left(\frac{dp}{dz}+\rho g\right)\left[2.089\times10^5\right] \qquad \left(\frac{dp}{dz}+\rho g\right)=-1524\ \text{N/m}^2\text{per metre}$$

(iii) The mean velocity u = Q/Cross Sectional Area = $\frac{0.05}{\pi\left(0.15^2-0.1^2\right)}=\frac{0.05}{0.03927}=1.273\ \text{m/s}$

$Re=\frac{\rho u D}{\mu}$ but as it is an annulus we cant use the diameter so how to check if its is laminar? My guess is it is based on radial gap. Knowing the critical value of Re is another matter.

(b) Proving where the maximum velocity occurs is another large task but a parabolic distribution reaches a maximum at mid point where r = 0.125 m

$$u=\frac{(1524)}{4\times0.25}\left[\frac{\left\{0.15^2-0.1^2\right\}}{\ln\left\{\frac{0.15}{0.1}\right\}}\ln\frac{0.125}{0.1}+0.1^2-0.125^2\right]=1.911\ \text{m/s}$$ NB this is 1.5 times the mean which is correct for a parabola.