MECHANICS OF SOLIDS
DYNAMICS

MOMENT OF INERTIA

On completion of this tutorial you should be able to

- Revise angular motion.
- Define and derive the moment of inertia of a body.
- Define radius of gyration.
- Examine Newton's second law in relation to rotating bodies.
- Define and use inertia torque.
- Define and use angular kinetic energy.
- Solve problems involving conversion of potential energy into kinetic energy.

It is assumed that the student is already familiar with the following concepts.

- Newton's Laws of Motion.
- The laws relating angular displacement, velocity and acceleration.
- The laws relating angular and linear motion.
- The forms of mechanical energy.

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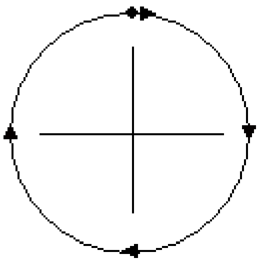
1. Revision of Angular Quantities

1.1 Angle θ



Angle has no units since it a ratio of arc length to radius. We use the names ***Revolution, Degree and Radian***.

Revolution
A point on a wheel that rotates one revolution traces out a circle. One revolution is the angle of rotation. This is a bit crude for use in calculations and we need smaller parts of the revolution.

Degrees
Traditionally we divide one revolution into 360 parts and call this a degree with symbol ^{o}.
1 revolution = 360^o

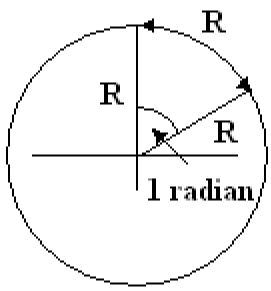
A single degree is not accurate enough for many applications so we divide a degree up into smaller parts called minutes. 1^o = 60 minutes or 60'

A minute can be divided up into even smaller bits called seconds and 1 minute = 60 seconds or 60".

In modern times we use decimals to express angles accurately so you are unlikely to use minutes and seconds.

Grads
In France, they divide the circle up into 400 parts and this is called a Grad. 1 revolution = 400 Grad. This makes a quarter of a circle 100 Grads whereas in degrees it would be 90^o.

Radian
In Engineering and Science, we use another measurement of angle called the Radian. This is defined as the angle created by placing a line of length 1 radius around the edge of the circle as shown. In mathematical words it is the angle subtended by an arc of length one radius. This angle is called the ***Radian***.



The circumference of a circle is 2πR. It follows that the number of radians that make a complete circle is:

$$\frac{2\pi R}{R} = 2\pi$$

There are 2π radians in one revolution so 360^o = 2π radian. 1 radian = 360/2π = 57.296^o

In the following work we will be using degrees and radian so it is very important that you make sure your calculator is set to the units that you are going to use. You might find a button labelled DRG on your calculator. Press this repeatedly until the display shows either D (for degrees) or R (for Radian) or if you are French, G (for Grad). On other calculators you might have to do this by using the mode button so read your instruction book. Also note that since one revolution is 2π radian and also 360^o we convert degrees into radian as follows.

$$\theta \text{ (radian)} = \text{degrees} \times 2\pi/360 = \text{degrees} \times \pi/180$$

1.2 Angular Velocity ω

The symbol for angular velocity is the lower case of the Greek letter Omega - ω.
If a body turns at constant speed, the angular velocity is the angle turned in 1 second. The angle turned in t seconds is then:

$$\theta = \omega t \text{ and so } \omega = \frac{\theta}{t} \text{rad}/s$$

If the body is speeding up or slowing down we may express the instantaneous angular velocity as the rate of change of angle per second.
In calculus form we can write:

$$\omega = \frac{d\theta}{dt}$$

In practical cases, angular velocity or speed is usually given in revolutions/second or revolutions/minute. When solving problems we nearly always have to convert this into radians/s. Since a circle (or revolution) is 2π radian we convert rev/s into rad/s by $\omega = 2\pi N$.

1.3 Angular Acceleration α

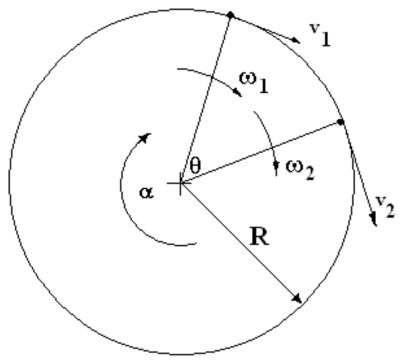
Angular acceleration (symbol α - alpha) occurs when a wheel speeds up or slows down. It is defined as the rate of change of angular velocity. Since angular velocity is the rate of change of angle we can also say that acceleration is the rate of change of the rate of change of angle.

If the wheel changes its velocity by $\Delta\omega$ in t seconds, the acceleration is $\alpha = \Delta\omega / t$ rad/s^2

In calculus form we can express this as:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

1.4 Link between Linear and Angular Quantities.



Consider a point moving on a circular path as shown.

In time t it rotates about the centre by angle θ and travels along the arc. The distance travelled on the circular path is the length of the arc 's' and:

$$\mathbf{s = R\theta}$$

The velocity along the circular path is $\mathbf{v = s/t = R\theta/t}$

$$\mathbf{v = R\,\omega}$$

Next suppose that the point accelerates from angular velocity ω_1 to ω_2. The velocity along the curve also changes from v_1 to v_2.
Angular acceleration = $\alpha = (\omega_2 - \omega_1)/t$
Substituting $\omega = v/R$

$$\alpha = \frac{\left(\frac{v_2}{R} - \frac{v_1}{R}\right)}{t} = \frac{a}{R} \text{ and } a = R\alpha$$

It is apparent that to change an angular quantity into a linear quantity all we have to do is multiply it by the radius.

1.5 Torque

When we rotate a wheel, we must apply torque to overcome the inertia and speed it up or slow it down. You should know that torque is a moment of force. A force applied to the axle of a wheel will not make it rotate (figure 1A). A force applied at a radius will (figure 1B).
The torque is F r (N m).

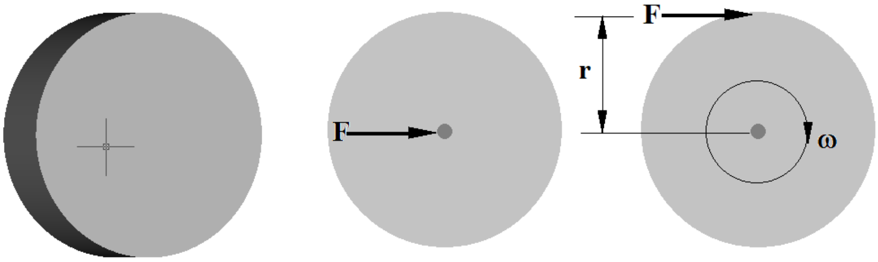


Figure A Figure B

2. Moment of Inertia I

The moment of inertia is that property of a body which makes it reluctant to speed up or slow down in a rotational manner. Clearly it is linked with mass (inertia) and in fact moment of inertia means second moment of mass. It is not only the mass that governs this reluctance but also the location of the mass. You should appreciate that a wheel with all the mass near the axle (fig. A) is easier to speed up than one with an equal mass spread over a larger diameter (fig.B).

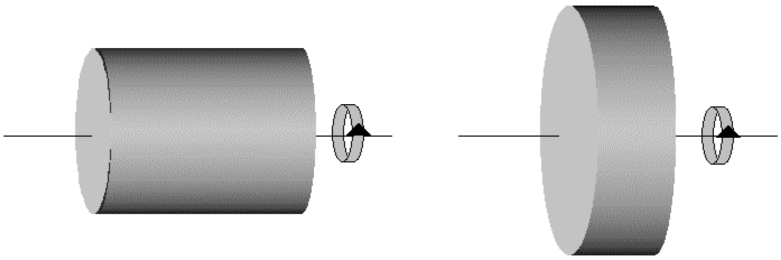
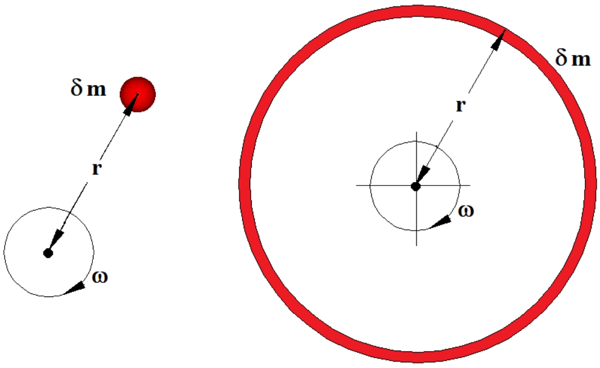


Figure A Figure B

2.1 Derivation

Consider the case where all the mass is rotating at one radius. This might be a small ball or a rim (like a bicycle wheel) as shown with mass δm at radius r. The angular velocity is ω rad/s.



If we multiply the mass by the radius we get the first moment of mass $r\,m$
If we multiply by the radius again we get the second moment of mass $r^2\,m$
This second moment is commonly called the **moment of inertia** and has a symbol **I**.

Unfortunately most rotating bodies do not have the mass concentrated at one radius and the moment of inertia is not calculated as easily as this.

2.1.1 Radius of Gyration K

All rotating machinery such as pumps, engines and turbines have a moment of inertia. The radius of gyration is the radius at which we consider the mass to rotate such that the moment of inertia is given by

$$\mathbf{I = M\,k^2}$$

M is the total mass and k is the radius of gyration.

The only problem with this approach is that the radius of gyration must be known and often this is deduced from tests on the machine.

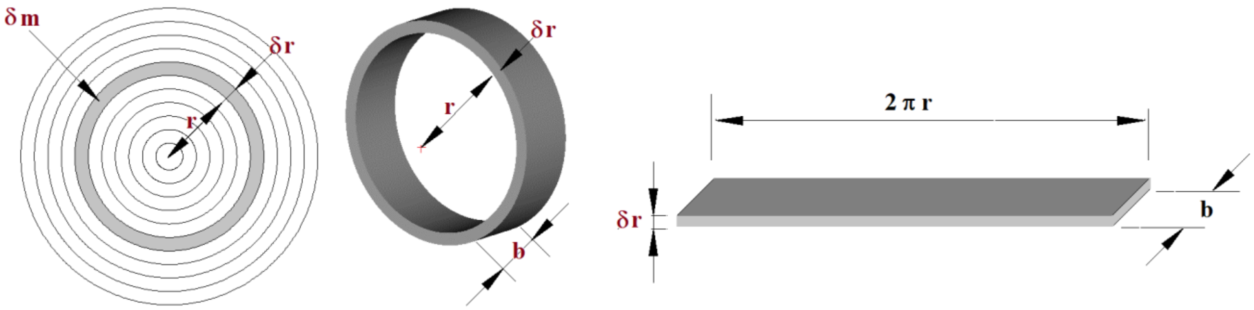
2.1.2 Plain Disc

Consider a plain disc and suppose it to be made up from many concentric rings or cylinders. Each cylinder is so thin that it may be considered as being at one radius r and the radial thickness is a tiny part of the radius δr. These are called elementary rings or cylinders.

If the mass of one ring is a small part of the total we denote it δm. The moment of inertia is a small part of the total and we denote it δI and this is given by $\delta I = r^2\,\delta m$. The total moment of inertia is the sum of all the separate small parts so we can write

$$\mathbf{I = \sum \delta I = \sum r^2\,\delta m}$$

The disc is b metres deep. Establish the formula for the mass of one ring.



The elementary thin cylinder if cut and unrolled would form a flat sheet as shown.

Length = circumference = $2\pi r$ depth depth = b thickness = δr

Volume = length × depth × thickness = $2\pi r\, b\, \delta r$

Change this to mass δm by multiplying by the density of the material ρ

Mass = $\delta m = \rho\, b\, 2\pi r\, \delta r$

If the mass is multiplied by the radius twice we get the moment of inertia δI.

$\delta I = \rho\, b\, 2\pi r\; r^2\, \delta r = \rho\, b\, 2\pi r^3\, \delta r$

As the radial thickness δr gets thinner and tends to zero, the equation becomes precise and we may replace the finite dimensions δ with the differential d.

$d I = \rho\, b\, 2\pi r^3\, dr$

The total moment of inertia is found by integration which is a way of summing all the rings that make up the disc.

$$I = \int_0^R 2\pi b\rho r^3 dr = 2\pi b\rho \int_0^R r^3 dr$$

Completing the integration and substituting the limits of r = 0 (the middle) and r = R (the outer radius) we get the following.

$$I = 2\pi b\rho \frac{[r^4]_0^R}{4} = 2\pi b\rho \frac{[R^4 - 0]}{4} = \pi b\rho \frac{R^4}{2}$$

Now consider the volume and mass of the disc. The volume of the plain disc is the area of a circle radius R times the depth b.

Volume = $\pi R^2 b$ Mass = volume × density = $\rho\pi R^2 b$

Examine the formula for I again.

$$I = \pi b\rho R^2 \frac{R^2}{2} = \text{Mass} \times \frac{R^2}{2}$$

$$\mathbf{I = \frac{MR^2}{2}} \text{ for a plain disc}$$

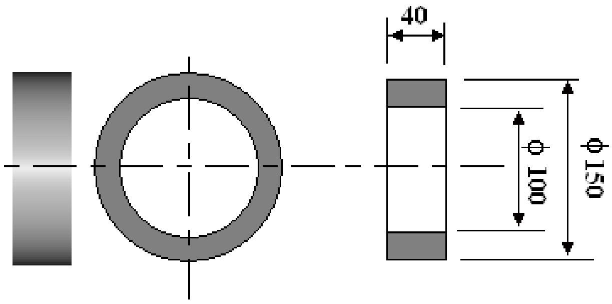
If we compare this to $I = Mk^2$ we deduce that the radius of gyration for a plan disc is

$$\mathbf{k = \sqrt{\frac{R^2}{2}} = 0.707R}$$

The effective radius at which the mass rotates is clearly not at the midpoint between the middle and the outside but nearer the edge.

WORKED EXAMPLE No. 1

Calculate the moment of inertia and radius of gyration for the ring shown made from steel of density 7 830 kg/m^3. The dimensions are in mm.



SOLUTION

First find I for the outer diameter.

$$I_1 = \pi b \rho \frac{R^4}{2} = \pi(0.04)\ (7\ 830) \times \frac{0.075^4}{2} = 0.01557 \text{ kg m}^2$$

Next find I for hole

$$I_2 = \pi b \rho \frac{r^4}{2} = \pi(7\ 830)(0.04) \text{ x } \frac{0.05^4}{2} = 0.003075 \text{ kg m}^2$$

Subtract $I = I_1 - I_2 = 0.0124915$ kg m^2

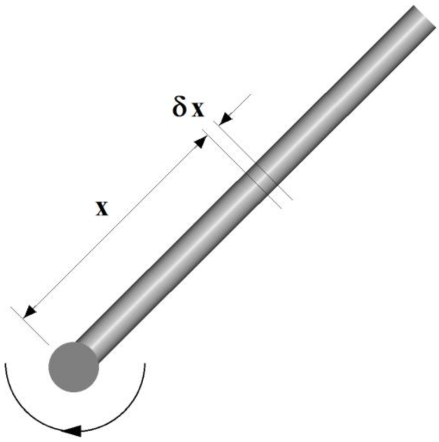
$I = Mk^2$ $M = \pi b\rho(R^2 - r^2) = \pi(0.04)(7\ 830)(\ 0.075^2 - 0.05^2) = 3.075$ kg

$$k = \sqrt{\frac{0.0125}{3.075}} = 0.064 \text{ m or } 64 \text{ m}$$

WORKED EXAMPLE No. 2

Show that the radius of gyration for a uniform rod of length L rotating about its end is 0.577L from that end.

SOLUTION



The rod has a uniform cross section A. Consider a small length δx. The mass of this element is

$$\delta m = \rho A \delta x$$

The moment of inertia is

$$\delta I = \delta m\, x^2 = \rho A x^2 \delta x$$

In the limit as δx gets smaller and smaller we may use the differential dx and integrate to find I.

$$I = \int_0^L \rho A x^2 \delta x = \rho A \int_0^L x^2 \delta x = \rho A \left[\frac{x^3}{3}\right]_0^L = \frac{\rho A L^3}{3}$$

The mass of the rod is

$$M = \rho A\, L \qquad I = M\frac{L^2}{3}$$

Using the radius of gyration

$$I = Mk^2$$

It follows that

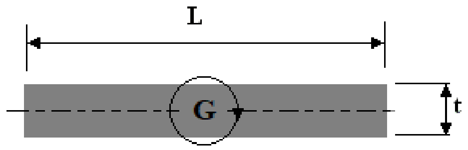
$$k^2 = \frac{L^2}{3} \quad k = \sqrt{\frac{L^2}{3}} = 0.577L$$

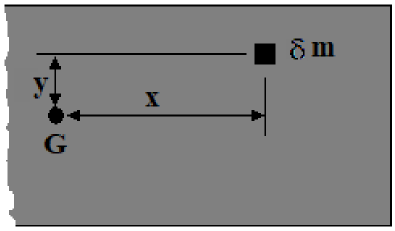
WORKED EXAMPLE No. 3

Show that the moment of inertia for a uniform rectangular bar of length L and cross sectional area A and width t rotating about its centre of gravity is

$$I_g = \frac{M(L^2 + t^2)}{12}$$

SOLUTION





Consider a small mass δm at coordinates x and y from the centre of gravity.

The moment of inertia about the y axis is $\delta m\, x^2$ and about the x axis it is $\delta m\, y^2$

$$I_x = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho A x^2 dx = \rho A \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \rho A \left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}} = \rho A \frac{L^3}{12} = \frac{ML^2}{12}$$

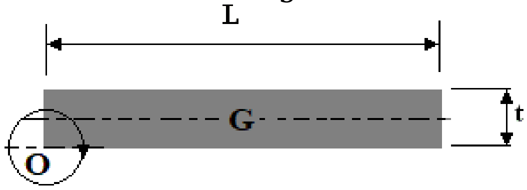
$$I_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \rho A x y^2 dy = \rho A \int_{-\frac{t}{2}}^{\frac{t}{2}} y^2 dy = \rho A \left[\frac{y^3}{3}\right]_{-\frac{t}{2}}^{\frac{t}{2}} = \rho A \frac{t^3}{12} = \frac{Mt^2}{12}$$

$$I_G = I_x + I_y = \frac{M(L^2 + t^2)}{12}$$

WORKED EXAMPLE No. 4

Show that the moment of inertia for a uniform rod of length L and thickness t rotating about its corner as shown is

$$I_O = \frac{M(L^2 + t^2)}{3}$$



SOLUTION

$$I_G = \frac{M(L^2 + t^2)}{12}$$

To find I_O we need the parallel axis theorem.

$$I_O = I_G + M(OG)^2$$

Pythagoras gives

$$(OG)^2 = \left(\frac{L}{2}\right)^2 + \left(\frac{t}{2}\right)^2 = \frac{(L^2 + t^2)}{4}$$

$$I_O = \frac{M(L^2 + t^2)}{12} + \frac{M(L^2 + t^2)}{4} = \frac{M(L^2 + t^2)}{3}$$

2.2 Angular Kinetic Energy

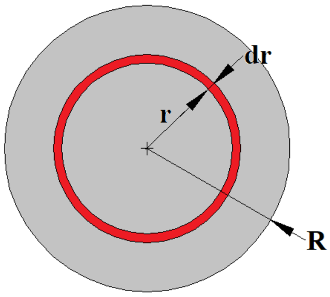
You should already know that linear kinetic energy is given by the formula

$$\text{K.E.} = \frac{mv^2}{2}$$

It requires energy to accelerate a wheel up to speed so rotating bodies also possess kinetic energy and the formula is

$$\text{K.E.} = \frac{m\omega^2}{2}$$

2.2.1 Derivation



Consider again a disc made of elementary rings. If a point rotates about a centre with angular velocity ω rad/s, at radius r, the velocity of the point along the circle is v m/s and it is related to ω by v = ωr.

The mass of the ring is δm.

$$\delta m = 2\pi r\, dr\, b\, \rho$$

Kinetic Energy of ring

$$KE = \frac{\delta m\, v^2}{2} = \frac{2\pi r\, dr\, b\, \rho v^2}{2} = \pi r\, b\, \rho v^2 dr$$

Convert linear velocity to angular velocity v = ω r

$$KE = \pi\, r\, b\, \rho\, \omega^2 r^2 dr = \pi\, b\, \rho\, \omega^2 r^3 dr$$

The total kinetic energy for the disc is found by integration so

$$KE = \int_0^R \pi\, b\, \rho\, \omega^2 r^3 dr = \pi\, b\, \rho\, \omega^2 \int_0^R r^3 dr = \pi\, b\, \rho\, \omega^2 \frac{R^4}{4} = \pi\, R^2 b\, \rho\, \omega^2 \frac{R^2}{4}$$

The mass of the disc is $M = \pi\, R^2\, b\, \rho$

$$KE = M\, \omega^2 \frac{R^2}{4}$$

The moment of inertia is:

$$I = M\, \frac{R^2}{2}$$

Hence:

$$\mathbf{KE = \frac{I\, \omega^2}{2}}$$

WORKED EXAMPLE No. 5

Find the kinetic energy of a wheel rotating at 4 rad/s given the mass is 3 kg and the radius of gyration is 0.2 m.

SOLUTION

$I = M\, k^2 = 3 \times 0.2^2 = 0.12 \text{ kg m}^2$

$\text{K. E.} = I\, \omega^2/2 = 0.12 \times 4^2/2 = 0.96$ Joules

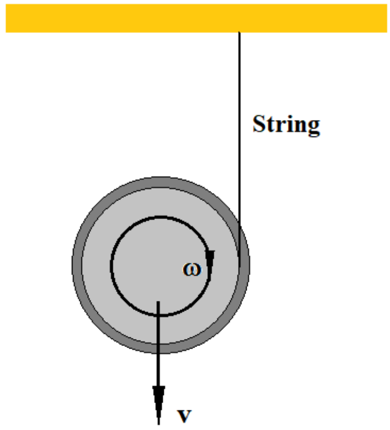
WORKED EXAMPLE No. 6

Find the kinetic energy of a wheel rotating at 4 rad/s given the mass is 3 kg and the radius of gyration is 0.2 m.

SOLUTION

$I = M k^2 = 3 \times 0.2^2 = 0.12$ kg m^2

K.E. = $I \omega^2/2 = 0.12 \times 4^2/2 = 0.96$ Joules



WORKED EXAMPLE No. 7

The wheel shown has a mass of 0.5 kg and a radius of gyration of 0.02 m. The radius of the drum is 0.01 m.

Calculate the linear and angular velocity of the wheel after it falls 0.6 m.

SOLUTION

As the wheel falls it will lose potential energy and gain two forms of kinetic energy because it has a velocity down and the string makes it spin. First calculate the change in potential Energy when it falls 0.6 m.

P.E. = $mg z = 0.5 \times 9.81 \times 0.6 = 2.943$ J

Next formulate the kinetic energy gained.

Linear K.E. = $mv^2/2$

Angular kinetic energy = $I \omega^2/2$

Calculate the moment of inertia $I = Mk^2 = 0.5 \times 0.02^2 = 200 \times 10^{-6}$ kg m^2

Equate the P.E. to the K. E.

$$2.943 = \frac{mv^2}{2} + \frac{I\omega^2}{2} = \frac{0.5v^2}{2} + \frac{200 \text{ x } 10^{-6}\omega^2}{2}$$

Substitute $v = \omega r$

$$2.943 = \frac{0.5\omega^2 r^2}{2} + \frac{200 \text{ x } 10^{-6}\omega^2}{2} = \frac{0.5\omega^2 \text{x } 0.01^2}{2} + \frac{200 \text{ x } 10^{-6}\omega^2}{2}$$

$$2.943 = 25 \text{ x } 10^{-6}\omega^2 + 100 \text{ x } 10^{-6}\omega^2 = 125 \text{ x } 10^{-6}\omega^2$$

$$\omega^2 = 23\,544 \qquad \omega = 153.44 \text{ rad/s}$$

Now calculate $v = \omega r = 153.44 \times 0.01 = 1.53$ m/s

SELF ASSESSMENT EXERCISE No. 1

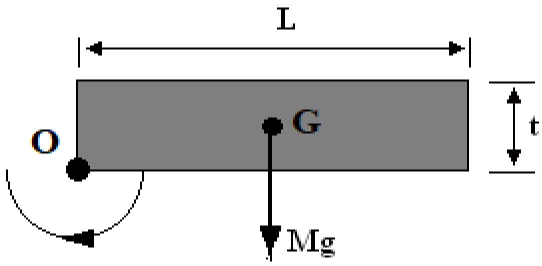
1. A cylinder has a mass of 1 kg, outer radius of 0.05 m and radius of gyration 0.03 m. It is allowed to roll down an inclined plane until it has changed its height by 0.6 m. Assuming it rolls with no energy loss, calculate its linear and angular velocity at this point.

2. A cylinder has a mass of 3 kg, outer radius of 0.2 m and radius of gyration 0.15 m. It is allowed to roll down an inclined plane until it has changed its height by 2 m. Assuming it rolls with no energy loss, calculate its linear and angular velocity at this point.

   (Answers 25.06 rad/s and 5.01 m/s)

3. A uniform body of mass M has a rectangular cross section and it is supported in the horizontal position shown. The body is released. Show that the initial acceleration at the moment of release is given by the expression

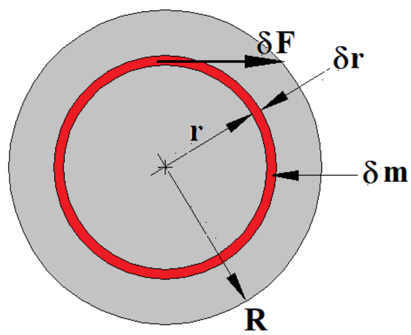
$$\alpha = \frac{3gL}{2(L^2 + t^2)}$$



3. *Inertia Torque*

This is the torque required to accelerate a wheel by overcoming the inertia of the wheel.

Consider a plain disc again and an elementary ring with a mass δm at radius r. Let a force δF be applied tangential to accelerate the ring.



The acceleration of the ring along a tangent is 'a'

Newton's 2nd Law for linear motion states that
Force = mass × acceleration.
In this case $\delta F = \delta m\, a$

Change to angular acceleration using the relationship $a = \alpha r$

$\delta F = \delta m\, \alpha r = r\, \delta m\, \alpha$ Multiply both sides by r.

$r\, \delta F = r^2 \delta m\, \alpha$

The expression r δF is the torque on the ring and we will denote it as δT.

The term $r^2\delta m$ is the moment of inertia for the elementary ring and this is a small part of the total moment of inertia for the disc so denote it δI

The expression becomes $\delta T = \delta I\, \alpha$

The plain disc is made up of many such rings so the torque needed to accelerate the whole disc at α rad/s^2 is found by summing them.

$$T = \sum_0^T \delta T = \alpha \sum_0^I \delta I = I\alpha$$

Hence in order to accelerate a disc with angular acceleration α, a torque T is required and the relationship is

$$\mathbf{T = I\alpha}$$

This is Newton's 2nd law written in angular terms and it applies to any shape, not just plain discs.

Remember that the formula for the moment of inertia is as follows.

For any rotating body in general $I = Mk^2$
k is the radius of gyration.
For a plain disc radius R, $I = MR^2/2$

For an annular ring of inside radius R_i and outside radius R_o

$$I = m\frac{(R_o^2 + R_i^2)}{2}$$

WORKED EXAMPLE No. 8

A solid wheel has a mass of 5 kg and outer radius of 200 mm. Calculate the torque required to accelerate it from rest to 1 500 rev/min in 10 seconds.

SOLUTION

The initial angular velocity is zero $\omega_1 = 0$
The final angular velocity is found by converting 1 500 rev/min

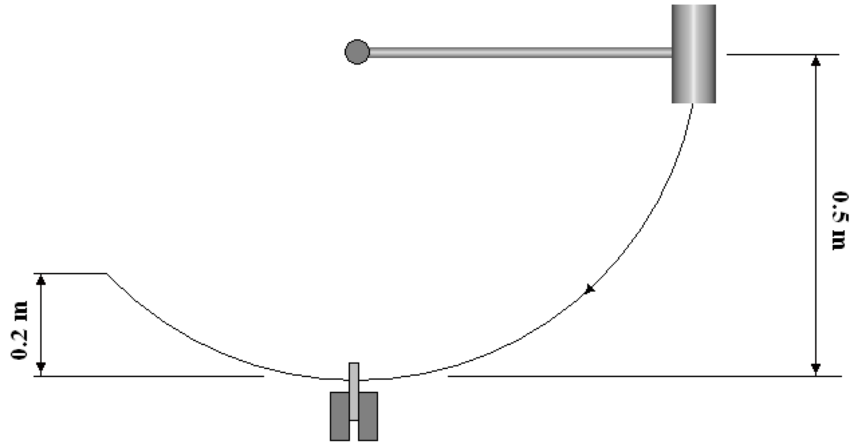
$$\omega_2 = \frac{1\,500}{60} \times 2\pi = 157.1\frac{\text{rad}}{\text{s}} \qquad \alpha = \frac{\omega_2 - \omega_1}{t} = \frac{157.1 - 0}{10} = 15.71\ \text{rad/s}^2$$

$$I = \frac{MR^2}{2} = \frac{5 \times 0.2^2}{2} = 0.1\ \text{kg m}^2 \qquad T = I\alpha = 0.1 \times 15.71 = 1.571\ \text{N}$$

WORKED EXAMPLE No. 9

A simple impact tester consists of a hammer of mass 0.6 kg on a pivoted light rod 0.5 m long as shown. The hammer is raised to the horizontal position and allowed to swing down hitting the test sample in a vice as shown. The hammer continues in its swing to a height of 0.2 m on the other side. Determine

i. The velocity of the hammer when it strikes the sample.
ii. The energy absorbed in the impact.
iii. The angular velocity at the bottom of the swing.



SOLUTION

The initial potential energy of the hammer = mg z_1 = 0.6 × 9.81 × 0.5 = 2.943 J

The Kinetic energy at the lowest point will be the same if none is lost so equate them.

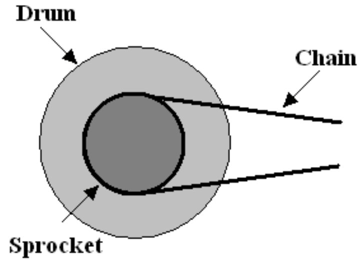
KE = $mv^2/2$ = 2.943 J   0.6 $v^2/2$ = 2.943   v = 3.13 m/s

The potential energy at the end of the swing = mg z_2 = 0.6 × 9.81 × 0.2 = 1.177 J

Energy lost in the impact = 2.943 - 1.177 = 1.766 J

The angular velocity of the hammer at the bottom of the swing = ω = v/r = 3.13/0.5 = 6.26 rad/s

WORKED EXAMPLE No. 10



A drum has a mass of 200 kg and radius of gyration of 0.8 m. It is rotated by a chain drive. The sprocket on the drum has an effective diameter of 0.3 m. The force in the chain is a constant value 120 N.
Calculate the time required to accelerate it up to 20 rev/min.

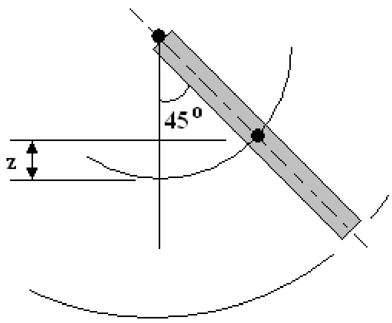
SOLUTION

$I = Mk^2 = 200 \times 0.8^2 = 128$ kg m^2. $T = F\,R = 120 \times 0.15 = 18$ N m

$$T = I\alpha \quad \alpha = \frac{T}{I} = \frac{18}{128} = 0.1406 \text{ rad/s}^2 \quad \omega_1 = 0 \text{ rad/s} \quad \omega_2 = \frac{20}{60} \times 2\pi = 2.094 \text{ rad/s}$$

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{2.094}{t} = 0.1406 \text{ rad/s}^2 \quad t = \frac{2.094}{0.1406} = 14.9 \text{ s}$$

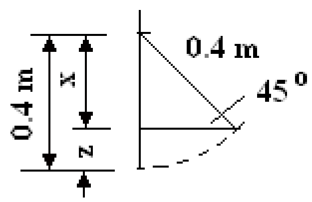
WORKED EXAMPLE No. 11



A rod of mass 2 kg has a uniform cross section and is 0.8 m long. It is pivoted at one end. The rod is turned in the vertical plane until it makes an angle of 45^o with the vertical as shown.

The rod is released and allowed to swing freely. Determine the velocity of the tip as it passes through the vertical

SOLUTION



The rod is uniform so the centre of gravity is at the centre. Calculate the change in height of the centre of gravity z. The geometry is shown below.

$x = 0.4 \sin 45^o = 0.2828$ m. $z = 0.4 - 0.2828 = 0.117$ m

The potential energy lost is $mgz = 2 \times 9.81 \times 0.117 = 2.299$ Joules.

Kinetic Energy gained = $I\omega^2/2$

For a plain rod $I = Mk^2$ and $k = 0.577\,L$ (worked example No.1)

$k = 0.577 \times 0.8 = 0.4616$ m $I = 2 \times 0.4616^2 = 0.426$ kg m^2

Equate KE and PE

$$2.299 = \frac{I\omega^2}{2} = \frac{0.426\omega^2}{2} \quad \omega^2 = \frac{2.299 \times 2}{0.426} = 10.79 \quad \omega = 3.28 \text{ rad/s}$$

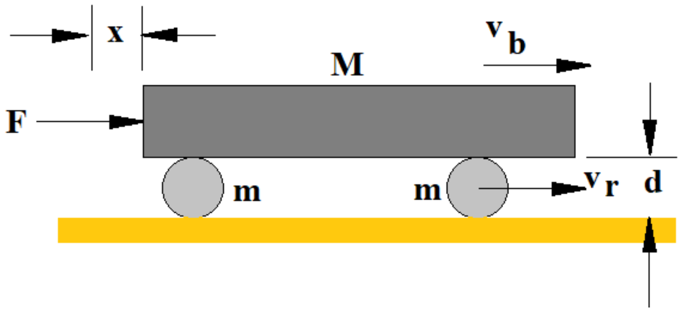
The velocity at the tip of the rod will be $v = \omega r = 3.28 \times 0.8 = 2.63$ m/s

WORKED EXAMPLE No. 12

A block of mass M rests on two rollers of mass m and diameter d. A force F is applied to the block as shown. Assuming no slip at any point, show that the acceleration of the block is given by the expression:

$$a = \frac{Fx}{\left\{\frac{M}{2} + m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}} \times \frac{1}{2x} = \frac{F}{\left\{M + 2m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}}$$

SOLUTION



The key to this type of problem is to recognise that the motion of the block is double the motion of the roller centres. If the roller rotates one revolution they advance one circumference relative to the ground and the block advances one circumference relative to the roller. The block advances two circumferences relative to the ground.
The radius of gyration for the rollers is k.

Let the block advance a distance x. The work done is W = F x
This is turned into kinetic energy of the block and rollers.

$$\text{K. E. of block} = \frac{M v_b^2}{2} \qquad v_b = \text{velocity of the block}$$

$$\text{K. E. of rollers} = 2\frac{I\,\omega^2}{2} + 2\frac{m{v_r}^2}{2} \qquad v_r = \text{velocity of the rollers}$$

$$Fx = \frac{M v_b^2}{2} + 2\frac{I\,\omega^2}{2} + 2\frac{m{v_r}^2}{2} \qquad v_r = \text{velocity of the rollers}$$

$$Fx = \frac{M v_b^2}{2} + I\,\omega^2 + m{v_r}^2 \qquad \omega = \frac{v_r}{\text{radius}} = \frac{2v_r}{d}$$

$$Fx = \frac{M v_b^2}{2} + I\left(\frac{2v_r}{d}\right)^2 + m{v_r}^2$$

$$Fx = \frac{M v_b^2}{2} + \frac{4I{v_r}^2}{d^2} + m{v_r}^2 \quad \text{but } v_r = \frac{v_b}{2} \text{ and } I = mk^2$$

$$Fx = \frac{M v_b^2}{2} + \frac{4mk^2{v_b}^2}{4d^2} + \frac{m{v_b}^2}{4}$$

$$Fx = \frac{M v_b^2}{2} + \frac{mk^2{v_b}^2}{d^2} + \frac{m{v_b}^2}{4}$$

$$Fx = \left\{\frac{M}{2} + m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}{v_b}^2$$

$$v_b{}^2 = \frac{Fx}{\left\{\frac{M}{2} + m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}}$$

If the velocity goes 0 to v in t seconds the acceleration is a = v/t

The distance travelled by the block is x = vt/2 hence a = v^2/2x

$$a = \frac{Fx}{\left\{\frac{M}{2} + m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}} \times \frac{1}{2x} = \frac{F}{\left\{M + 2m\left(\frac{k^2}{d^2} + \frac{1}{4}\right)\right\}}$$

SELF ASSESSMENT EXERCISE No. 2

1. A pendulum has a mass of 20 kg. It is fixed on the end of a rod of mass 10 kg and 1.8 m long. The pendulum is raised through 45^o and allowed to swing down and strike the specimen. After impact it swings up 20^o on the other side.

   Determine the following.

   i. The velocity just before impact. (3.33 m/s)
   ii. The energy absorbed by the impact. (102.7 J)

   Hints for solution.
   Calculate the P.E. of the hammer and rod separately assuming the centre of gravity of the rod is at the middle.

   Calculate the moment of inertia of the hammer with a k = 1.8 and for the rod with k = 0.577L. Equate PE to angular KE and then convert angular velocity to linear velocity.

2. An experiment is performed to find the moment of inertia of a flywheel as follows. A mass is attached by a string to the axle which has a radius of 37.5 mm. The mass is adjusted until its weight is just sufficient to overcome frictional resistance and rotate the flywheel without acceleration. This mass is 2 kg. Another 2.5 kg is added and the mass is allowed to fall under the action of gravity and measurements show that it takes 5 s to fall 1.5 m.

   Determine the following.

   i. The angular acceleration. (3.2 rad/s^2)

   ii. The moment of inertia of the flywheel. (0.281 kg m^2)

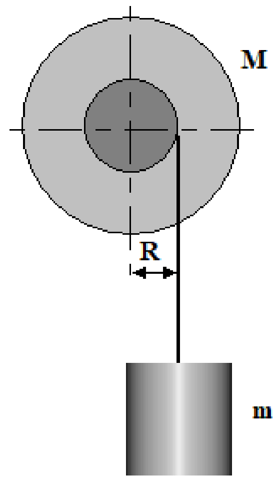
4. *The Use of Free Body Diagrams and D'Alembert's Principle*

In complex systems we must isolate different sections and apply the principles of the free body diagram and force balance to work out unknown forces. This is best illustrated with a worked example.

WORKED EXAMPLE No. 13

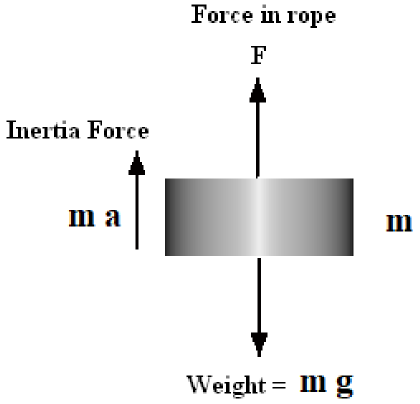
A drum with a mass M and radius of gyration k is accelerated by a falling mass m on the end of a rope pulling on a pulley of radius R as shown.

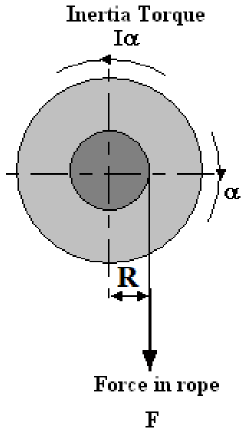
Derive a formula for the angular acceleration of the drum and the linear acceleration of the falling mass.



SOLUTION

First draw a free body diagram for the falling mass showing all the forces acting on it. The mass accelerates downwards at a m/s^2. This means the inertia must be overcome.





The inertia force always opposes motion so it acts upwards. Balancing the forces we have:

$$F = mg - ma = m(g - a)$$

Note if the rope was cut F = 0 and a = g and it would fall freely accelerating under the action of gravity. Also note that for convenience downwards acceleration is taken as positive. If use up as positive then g = -9.81 m/s^2.

Next do the same for the drum. $I = Mk^2$

The torque due to the force in the rope is FR. This acts clockwise. The inertia torque is Iα and opposes acceleration and is anticlockwise. Balancing the torques we have:

$$FR = I\alpha \text{ so } F = I\alpha/R$$

Now equate the force in the rope. $I\alpha/R = m(g - a)$ and we know that $a = \alpha R$ so:

$$\frac{I\alpha}{R} = m(g - \alpha R) = mg - m\alpha R \qquad \frac{I\alpha}{R} + m\alpha R = mg$$

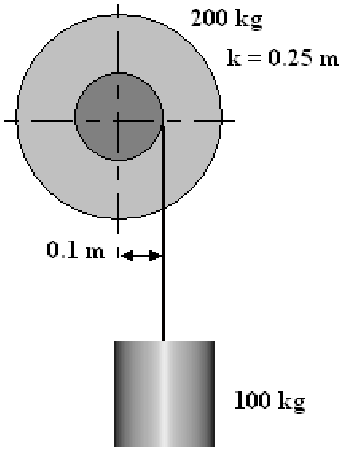
$$\left(\frac{I}{R} + m\,R\right)\alpha = mg \qquad \alpha = \frac{mg}{\frac{I}{R} + m\,R}$$

Since a = αr then:

$$a = \frac{mg}{\frac{I}{R} + m}$$

SELF ASSESSMENT EXERCISE No. 3

1. A drum is accelerated under the action of a falling weight as shown. Calculate the angular acceleration of the drum and the linear acceleration of the weight.



(Answers 7.27 rad/s^2 and 0.727 m/s^2)

2. The same system as in question 1 is accelerated upwards 0.5 m/s^2. What is the applied torque? (165.5 Nm)

3. An experiment is performed to find the moment of inertia of a flywheel as follows. A mass is attached by a string to the axle which has a radius of 37.5 mm. The mass is adjusted until its weight is just sufficient to overcome frictional resistance and rotate the flywheel without acceleration. This mass is 2 kg. Another 2.5 kg is added and the mass is allowed to fall under the action of gravity and measurements show that it takes 5 s to fall 1.5 m. Determine the following.

   i. The angular acceleration. (3.2 rad/s^2)
   ii. The moment of inertia of the flywheel. (0.281 kg m^2)

4. A drum which is 300 mm in diameter has a moment of inertia of 200 kg m$_2$. It revolves at 2 rev/s. A braking torque of 285.8 N m is applied to it producing uniform deceleration. Determine the following.

   i. The initial kinetic energy stored in the drum (15 791 J)
   ii. The angular deceleration of the drum. (1.43 rad/s^2).
   iii. The time taken to for the drum to stop. (8.8 s)

SELF ASSESSMENT No. 4

These are degree level questions

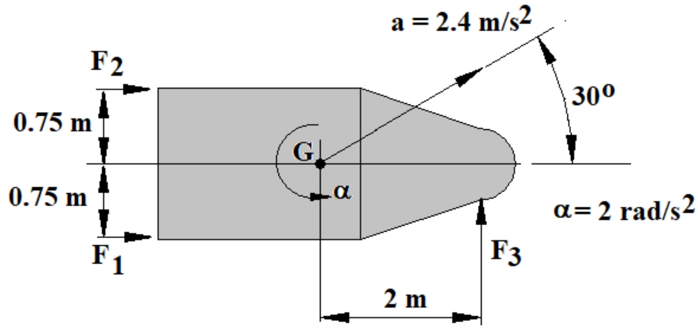
1. A lift has a mass of 1 200 kg. It is raised by a rope passing around a winding drum and a counterbalance mass of 1 000 kg hangs down on the other end. The drum has a radius of 1 m and a moment of inertia of 500 kg m2.

   During operation, the lift is accelerated upwards from rest at a rate of 2 m/s2 for 2 seconds. The lift then rises at constant velocity for another 10 seconds and then the drive torque is removed from the drum shaft and the lift coasts to a halt.

   Determine the following.

   i. The maximum and minimum force in rope during this period. (14.17 kN and 9.81 kN)
   ii. The torque applied to the drum during the acceleration period. (7.362 kNm)
   iii. The rate of deceleration. (1.16 m/s2)
   iv. The distance moved by the lift during the journey. (50.9 m)

2. The diagram shows a schematic of a space module orbiting in a gravity free zone. Rockets are attached at points and exert forces F_1, F_2 and F_3 as shown all acting in the same plane as the centre of mass G. The forces produce a linear acceleration a and angular acceleration α at G. The module has a mass of 1000 kg and moment of inertia 800 kg m^2 about G. Determine the three forces F_1, F_2 and F_3. (1 573 N, 505 N and 1 200 N)



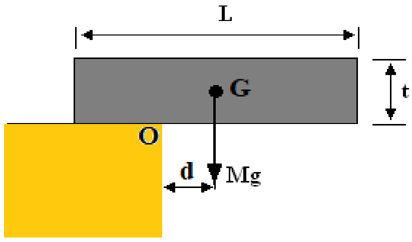
3. The uniform block of mass M is released from the position shown. If no slip occurs between the block and the support at point O, show that the angular acceleration of the block is given by

$$\alpha = \frac{6g(2d\cos\theta + t\sin\theta)}{L^2 + 12d^2 + 4t^2}$$

θ is the angle between the block upper surface and the horizontal when it is tilted.

The moment of inertia about the centre of gravity is

$$I_G = \frac{M(L^2 + t^2)}{12}$$



4. Two rollers of radius r, mass m and moment of inertia I support a truck of mass M as shown. Show that the acceleration of the block down the slope whilst it remains in contact with both rollers is given by.

$$\frac{g\sin\theta(M + m)}{M + \frac{m}{2} + \frac{I}{2r^2}}$$

There is no slip at the contact points.