SOLID MECHANICS
DYNAMICS
GEARED SYSTEMS

On completion of this short tutorial you should be able to do the following.

- ➢ Describe the different types of gear systems.
- ➢ Describe a simple gear train.
- ➢ Describe a compound gear rain.
- ➢ Describe three types of epicyclic gear boxes
- ➢ Solve gear box ratios.
- ➢ Calculate the input and outputs speeds and torques of gear boxes.
- ➢ Calculate the holding torque on gear box cases

Contents

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1. **Introduction**

A gear box is a device for converting the speed of a shaft from one speed to another. In the process the torque T is also changed. This can be done with pulley and chain drives but gears have advantages over these system. A good example is that of winch in which a motor with a high speed and low torque is geared down to turn the drum at a low speed with a large torque. Similarly, a marine engine may use a reduction gear box to reduce the speed of the engine to that of the propeller. Other examples are motor vehicles, lathes, drills and many more. The diagram shows a typical winch that has a reduction gear box built inside the drum.

This tutorial is not about the design of gears but it should be mentioned that there are many types of gears, each with their own advantages. Here are some examples.

WORM   SPUR   BEVEL   HELICAL   EPICYCLIC

Gears are wheels which mesh with each other through interlocking teeth. Rotation of one wheel produces rotation of the other with no slip between them. The shape of the gear teeth is important in order to produce a smooth transfer of the motion.

The design of the gear teeth also affects the relative position of one gear to another. For example bevelled gears allow the axis of one gear to be inclined to the axis of another. Worm gears convert the motion through 90° and so on. The design also affects the friction present in the transfer.

The helical gears run more quietly than most because there is always contact between them.

2.   *Gear Geometry*

The basic principle of a gear is that it is two levers acting against each other in turn to exert a force and hence torque from one to the other. In the ideal gear, the point of contact would always be a rolling contact. This is impossible to achieve in practice and many geometric shapes have been tried to come as close to this as possible. The one most universally used is the ***Involute*** although the ***Cycloid*** shape is just as good. This is not discussed here.
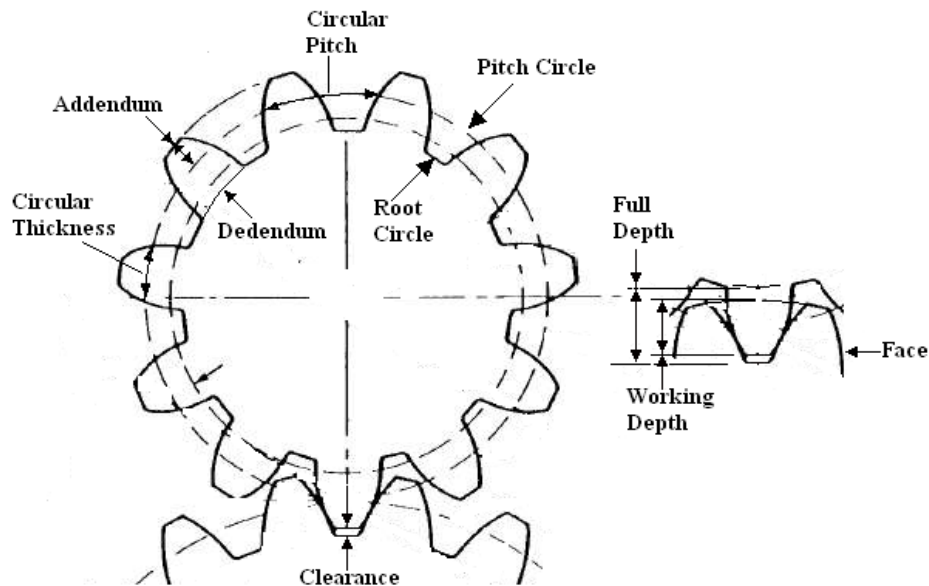
The diagram shows a spur gear but the same basic principles apply to spiral gears and bevel gears. Here are some of the definitions used with gears.

Pitch Circle – the effective diameter of the gear used in calculating the velocity ratio.

Circular Pitch - the distance between the teeth measured along the pitch circle. This must be the same both gears.

Diametral Pitch – the number of teeth ÷ pitch circle diameter (used only for dimensions in inches)

Module - pitch circle diameter ÷ the number of teeth (Used for dimensions in mm)
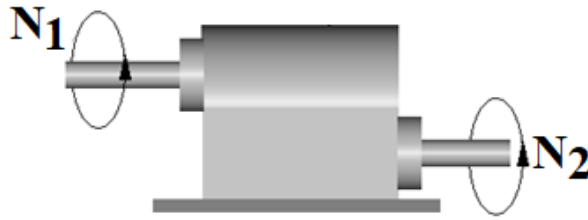
Addendum Circle – the outer diameter of the gear.

Root Circle or Dedendum Circle – the diameter at the root.

Addendum - radial distance from the pitch circle to the addendum circle.

Dedendum - radial distance from the pitch circle to the dedendum circle.

3. Basic Gear Box Theory

Consider a simple schematic of a gear box with an input and output shaft.

Velocity Ratio - Gear Box Ratio
The velocity ratio is more usually referred to as the gear ratio and this term is used here.

$$G.R. = \frac{\text{Input Speed}}{\text{Output Speed}} = \frac{N_1}{N_2}$$

N is usually in rev/min but the ratio is the same whatever units of speed are used. If angular velocity is used then

$$G.R. = \frac{\text{Input Speed}}{\text{Output Speed}} = \frac{\omega_1}{\omega_2}$$

Torque and Efficiency

The power transmitted by a torque T Nm applied to a shaft rotating at N rev/min is given by

$$S.P. = \frac{2\pi NT}{60}$$

In an ideal gear box, the input and output powers are the same so

$$\frac{2\pi N_1 T_1}{60} = \frac{2\pi N_2 T_2}{60} \qquad 2\pi T_1 = N_2 T_2 \qquad \frac{T_2}{T_1} = \frac{N_1}{N_2} = G.R.$$

It follows that if the speed is reduced, the torque is increased and vice versa. In a real gear box, power is lost through friction and the power output is smaller than the power input. The efficiency is defined as

$$\eta = \frac{\text{Power Out}}{\text{Power In}} = \frac{2\pi N_2 T_2 \times 60}{2\pi N_1 T_1 \times 60} = \frac{N_2 T_2}{N_1 T_1}$$

Because the torque in and out is different, a gear box has to be clamped in order to stop the case or body rotating. A holding torque T_3 must be applied to the body through the clamps.
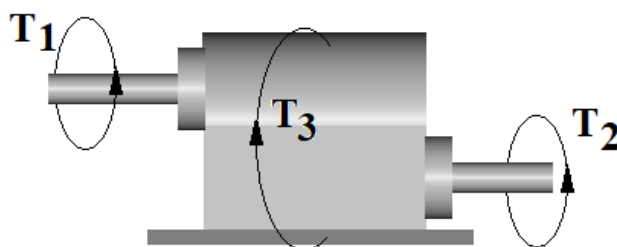
The total torque must add up to zero. $\mathbf{T_1 + T_2 + T_3 = 0}$
If we use a convention that anti-clockwise is positive and clockwise negative we can determine the holding torque. The direction of rotation of the output shaft depends on the design of the gear box.

WORKED EXAMPLE No. 1

1. A gear box has an input speed of 1 500 rev/min clockwise and an output speed of 300 rev/min anticlockwise. The input power is 20 kW and the efficiency is 70%. Determine the following.

i. The gear ratio
ii. The input torque.
iii. The output power.
iv. The output torque.
v. The holding torque.

SOLUTION

$$\text{G. R.} = \frac{\text{Input Speed}}{\text{Output Speed}} = \frac{N_1}{N_2} = \frac{1\,500}{300} = 5$$

$$\text{power In} = \frac{2\pi N_1 T_1}{60} \quad T_1 = \frac{60 \times \text{Power In}}{2\pi N_1} = \frac{60 \times 20\,000}{2\pi \times 1\,500} = 127.3 \text{ Nm (negative clockwise)}$$

$$\eta = 0.7 = \frac{\text{Power Out}}{\text{Power In}} \quad \text{Power Out} = 0.7 \times \text{Power In} = 0.7 \times 20 = 14 \text{ kW}$$

$$\text{Power Out} = \frac{2\pi N_2 T_2}{60} \quad T_2 = \frac{60 \times \text{Power Out}}{2\pi N_2} = \frac{60 \times 14\,000}{2\pi \times 300} = 445.6 \text{ Nm (positive A. C. W.)}$$

$$T_1 + T_2 + T_3 = 0$$

$$-127.3 + 445.6 + T_3 = 0$$

$$T_3 = 127.3 - 445.6 = -318.3 \text{ Nm (A. C. W.)}$$

SELF ASSESSMENT EXERCISE No. 1

1. A gear box has an input speed of 2000 rev/min clockwise and an output speed of 500 rev/min anticlockwise. The input power is 50 kW and the efficiency is 60%. Determine the following.

i. The input torque. (238.7 Nm)
ii. The output power. (30 kW)
iii. The output torque. (573 Nm)
iv. The holding torque. (334.3 Nm clockwise)

2. A gear box must produce an output power and torque of 40 kW and 60 Nm when the input shaft rotates at 1000 rev/min. Determine the following.

i. The gear ratio. (0.1571)
ii. The input power assuming an efficiency of 70% (57.14 kW)

4.   Types of Gear Trains

The meshing of two gears may be idealised as two smooth discs with their edges touching and no slip between them. This ideal diameter is called the Pitch Circle Diameter (PCD) of the gear.

4.1   Simple Gear Train.
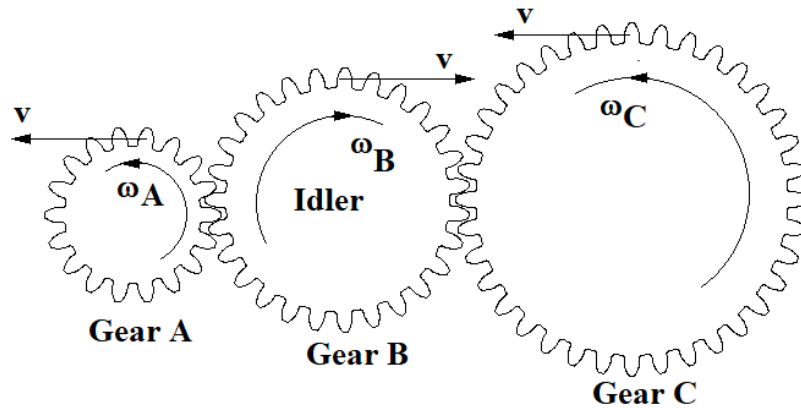
These are typically spur gears as shown. The direction of rotation is reversed from one gear to another. The only function of the idler gear is to change the direction of rotation. It has no affect on the gear ratio. The teeth on the gears must all be the same size so if gear A advances one tooth, so does B and C.

t = number of teeth on the gear.
D = Pitch circle diameter.
m = module = D/t and this must be the same for all gears otherwise they would not mesh.
$m = D_A/t_A = D_B/t_B = D_C/t_C$
$D_A = m\ t_A$ $D_B = m\ t_B$ $D_C = m\ t_C$
ω = angular velocity.
v = linear velocity on the circle.   v = ω D/2

The velocity v of any point on the circle must be the same for all the gears, otherwise they would be slipping. It follows that

$$\frac{\omega_A D_A}{2} = \frac{\omega_B D_B}{2} = \frac{\omega_C D_C}{2}$$

$$\omega_A D_A = \omega_B D_B = \omega_C D_C$$

$$\omega_A m\ t_A = \omega_B m\ t_B = \omega_C\ m\ t_C$$

$$\omega_A\ t_A = \omega_B\ t_B = \omega_C\ t_C$$

In terms of rev/min

$$N_A\ t_A = N_B\ t_B = N_C\ t_C$$

The gear ratio is defined as GR = Input speed/Output speed

If gear A is the input and gear C the output,

$$GR = \frac{N_A}{N_C} = \frac{t_C}{t_A}$$

WORKED EXAMPLE No. 2

A simple train has 3 gears. Gear A is the input and has 50 teeth. Gear C is the output and has 150 teeth. Gear A rotates at 1500 rev/min anticlockwise. Calculate the gear ratio and the output speed.

The input torques on A is 12 Nm and the efficiency is 75%. Calculate the output power and the holding torque.

SOLUTION

$$GR = \frac{N_A}{N_C} = \frac{t_C}{t_A} = \frac{150}{50} = 3$$

$$\frac{N_A}{N_C} = 3 \quad N_C = \frac{N_A}{3} = \frac{1\,500}{3} = 500 \text{ rev/min (ACW)}$$

$T_A = 12$ Nm

$$P(input) = \frac{2 \times \pi \times N_A T_A}{60} = \frac{2 \times \pi \times 1\,500 \times 12}{60} = 1\,885 \text{ W}$$

$$P(output) = P(Input) \times \eta = 1885 \times 0.75 = 1\,413.7 \text{ W}$$

$$T_C = \frac{60 \times P(output)}{2\pi \times 500} = \frac{60 \times 1\,413.7}{2\pi \times 500} = 27 \text{ Nm}$$

$$T_A + T_C + T_{hold} = 0$$

$$12 + 27 + T_{hold} = 0 \quad T_{hold} = -39 \text{ Nm (clockwise)}$$

SELF ASSESSMENT EXERCISE No. 2

1. A simple gear train has 2 spur gears. The input gear has 20 teeth and the output gear has 100 teeth. The input rotates at 2 000 rev/min clockwise. Calculate the gear ratio and the output speed. (5 and 400 rev/min anticlockwise)

2. The input torque is 15 Nm and the efficiency is 65%. Calculate the output power and the holding torque. (2 042 W and 33.75 Nm clockwise)

4.2   Compound Gears

Compound gears are simply a chain of simple gear trains with the input of the second being the output of the first. A chain of two pairs is shown below. Gear B is the output of the first pair and gear C is the input of the second pair. Gears B and C are locked to the same shaft and revolve at the same speed.
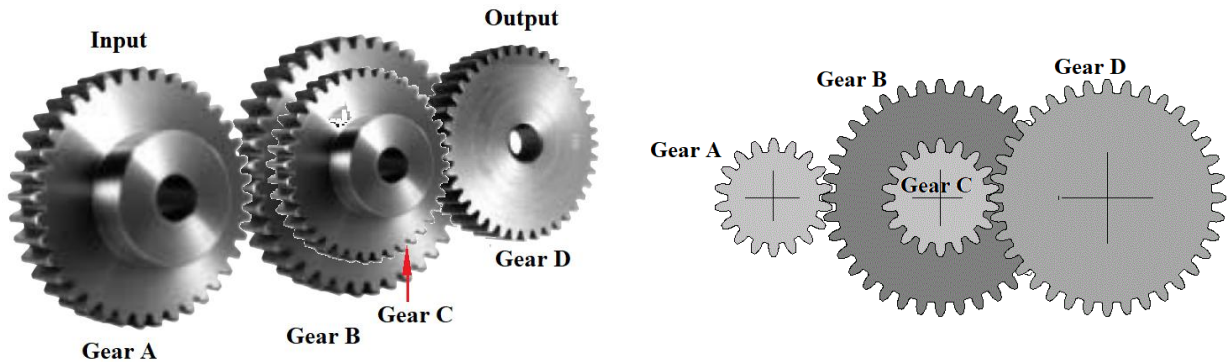
The velocity of each tooth on A and B are the same so $\omega_A t_A = \omega_B t_B$ as they are simple gears. Likewise for C and D, $\omega_C t_C = \omega_D t_D$.

$$\frac{\omega_A}{t_B} = \frac{\omega_B}{t_A} \text{ and } \frac{\omega_C}{t_D} = \frac{\omega_D}{t_C}$$

$$\omega_A = \frac{t_B \, \omega_B}{t_A} \text{ and } \omega_C = \frac{t_D \, \omega_D}{t_C}$$

$$\omega_A \omega_C = \frac{t_B \, \omega_B}{t_A} \times \frac{t_D \, \omega_D}{t_C} = \frac{t_B \, t_D \, \omega_B}{t_A t_C} \times \omega_B \omega_D$$

$$\frac{\omega_A \omega_C}{\omega_B \omega_D} = \frac{t_B \, t_D}{t_A t_C}$$

Since gears B and C are on the same shaft $\omega_B = \omega_C$

$$\frac{\omega_A}{\omega_D} = \frac{t_B \, t_D}{t_A t_C} = GR$$

Since $\omega = 2\pi N$ then the gear ratio may be written as

$$\frac{N_{in}}{N_{out}} = \frac{t_B \, t_D}{t_A t_C} = GR$$

Gears B and D are the driven gears. Gears A and C are the driver gears. It follows that

$$\text{Gear Ratio} = \frac{\text{Product of driven teeth}}{\text{Product of driving teeth}}$$

This rule applies regardless of how many pairs of gears there are.

WORKED EXAMPLE No. 3

Calculate the gear ratio for the compound chain shown below. If the input gear rotates clockwise, in which direction does the output rotate?
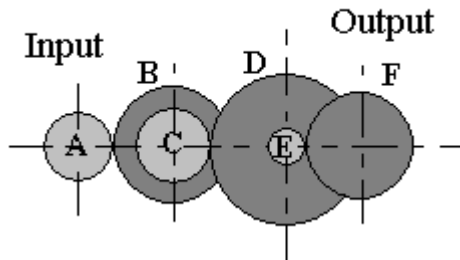
Gear A has 20 teeth
Gear B has 100 teeth
Gear C has 40 teeth
Gear D has 100 teeth
Gear E has 10 teeth
Gear F has 100 teeth

SOLUTION

The driving teeth are A, C and E.
The driven teeth are B, D and F

$$\text{Gear Ratio} = \frac{\text{Product of driven teeth}}{\text{Product of driving teeth}} = \frac{100 \times 100 \times 100}{20 \times 40 \times 10} = 125$$

Alternatively we can say there are three simple gear trains and work out the ratio for each.

First chain GR = 100/20 = 5
Second chain GR = 100/40 = 2.5
Third chain GR = 100/10 = 10
The overall ratio = 5 × 2.5 × 10 = 125

Each chain reverses the direction of rotation so if A is clockwise, B and C rotate anticlockwise so D and E rotate clockwise. The output gear F hence rotates anticlockwise.
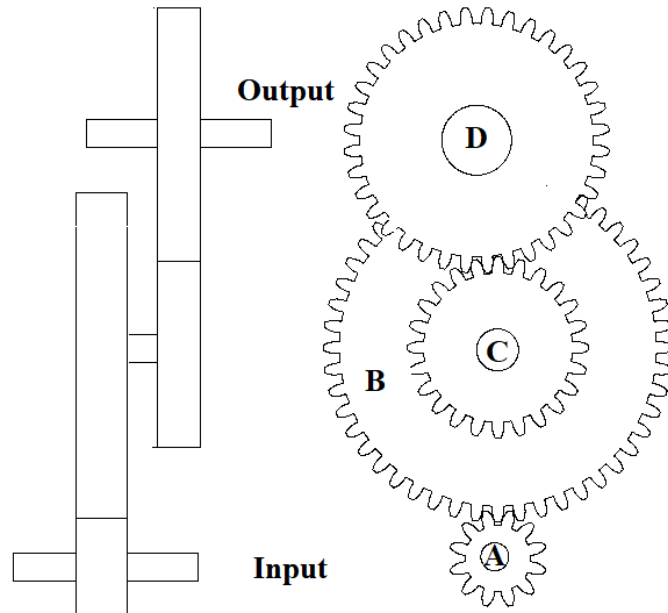
SELF ASSESSMENT EXERCISE No. 3

Gear A is the input and revolves at 1200 rev/min clockwise viewed from the left end. The input torque is 30 Nm and the efficiency is 70%.
Gear A has 12 teeth
Gear B has 48 teeth
Gear C has 24 teeth
Gear D has 36 teeth

Calculate the following.

i.   The output speed and its direction. (200 rev/min clockwise)
ii. The output power. (2 639 W)
iii. The fixing torque. (156 Nm anticlockwise)

4.3   Epicyclic Gears

Epicyclic means one gear revolving upon and around another. The design involves planet and sun gears as one orbits the other like a planet around the sun. Here is a picture of a typical gear box.

This design can produce large gear ratios in a small space and are used on a wide range of applications from marine gear boxes to electric screwdrivers.

Basic Theory
The diagram shows a gear B on the end of an arm A. Gear B meshes with gear C and revolves around it when the arm is rotated. B is called the planet gear and C the sun.
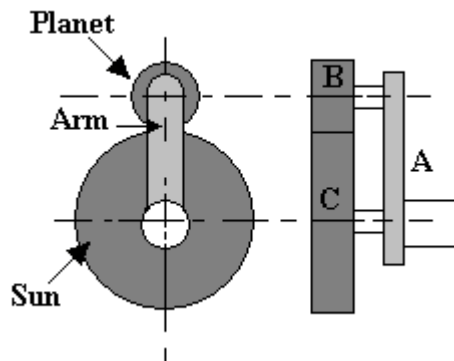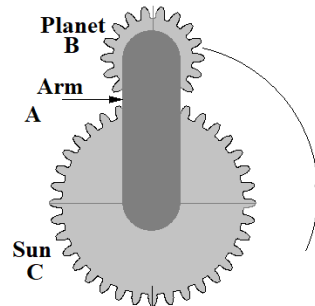
First consider what happens when the planet gear orbits the sun gear.
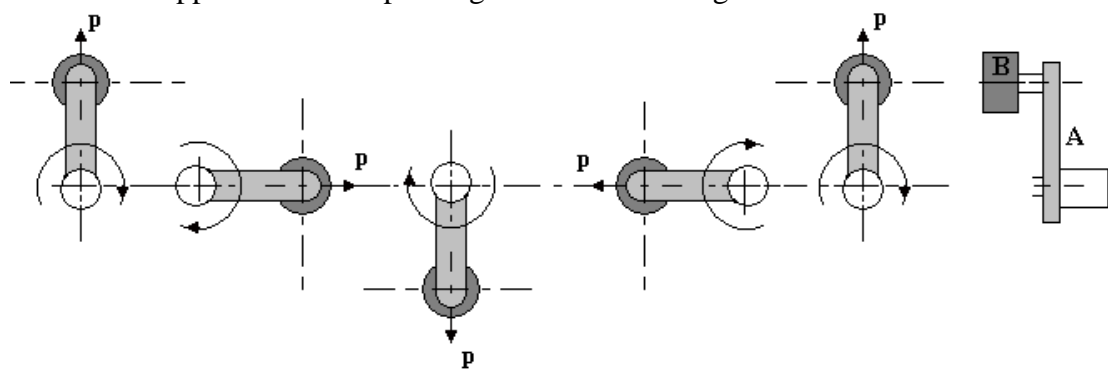
Observe point p and you will see that gear B also revolves once on its own axis. Any object orbiting around a centre must rotate once. Now consider that B is free to rotate on its shaft and meshes with C. Suppose the arm is held stationary and gear C is rotated once. B spins about its own centre and the number of revolutions it makes is the ratio t_C/t_B. B will rotate by this number for every complete revolution of C.

Now consider that C is unable to rotate and the arm A is revolved once. Gear B will revolve $t_C/t_B + 1$ because of the orbit. It is this extra rotation that causes confusion. One way to get round this is to imagine that the whole system is revolved once. Then identify the gear that is fixed and revolve it back one revolution. Work out the revolutions of the other gears and add them up. The following tabular method makes it easy.

Suppose gear C is fixed and the arm A makes one revolution. Determine how many revolutions the planet gear B makes.

Step 1 is to revolve everything once about the centre.
Step 2 identify that C should be fixed and rotate it backwards one revolution keeping the arm fixed as it should only do one revolution in total. Work out the revolutions of B.
Step 3 is simply add them up and we find the total revs of C is zero and for the arm is 1.

Step	Action	A	B	C
1	Revolve all once	1	1	1
2	Revolve C by -1 rev	0	$+ t_C/t_B$	-1
3	Add	1	$1+ t_C/t_B$	0

The number of revolutions made by B is $(1 + t_C/t_B)$. Note that if C revolves -1, then the direction of B is opposite so $+ t_C/t_B$

WORKED EXAMPLE No. 4

A simple epicyclic gear has a fixed sun gear with 100 teeth and a planet gear with 50 teeth. If the arm is revolved once, how many times does the planet gear revolve?
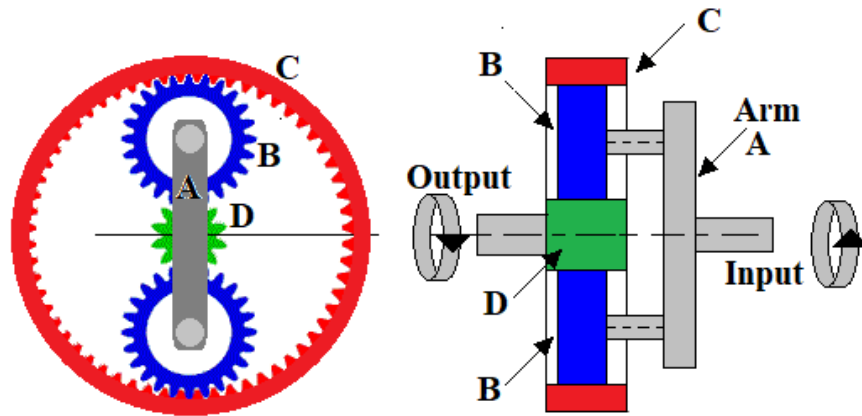
SOLUTION

| Step	Action	A	B	C
1	Revolve all once	1	1	1
2	Revolve C -1	0	+100/50	-1
3	Add	1	3	0

Gear B makes 3 revolutions for every one of the arm.

The design so far considered has no identifiable input and output. We need a design that puts an input and output shaft on the same axis. This can be done several ways.

METHOD 1

The arm is the input and gear D is the output. Gear C is a fixed internal gear and is normally part of the outer casing of the gear box. There are normally four planet gears and the arm takes the form of a cage carrying the shafts of the planet gears. Note that the planet gear and internal gear both rotate in the same direction.

External Gear
Reverse direction

Internal Gear
Same Direction
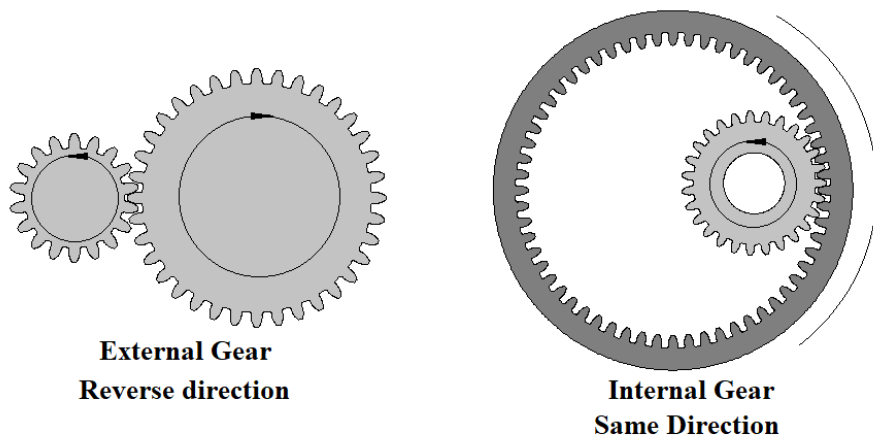
WORKED EXAMPLE No. 5

An epicyclic gear box has a fixed outer gear C with 240 teeth. The planet gears have 20 teeth. The input is the arm/cage A and the output is the sun gear D.

Calculate the number of teeth on the sun gear and the ratio of the gear box.

SOLUTION

The PCD of the outer gear must the sum of PCD of the sun plus twice the PCD of the planets so it follows that the number of teeth are related as follows.

$t_C = t_D + 2 \, t_B$
$240 = t_D + 2 \times 20$
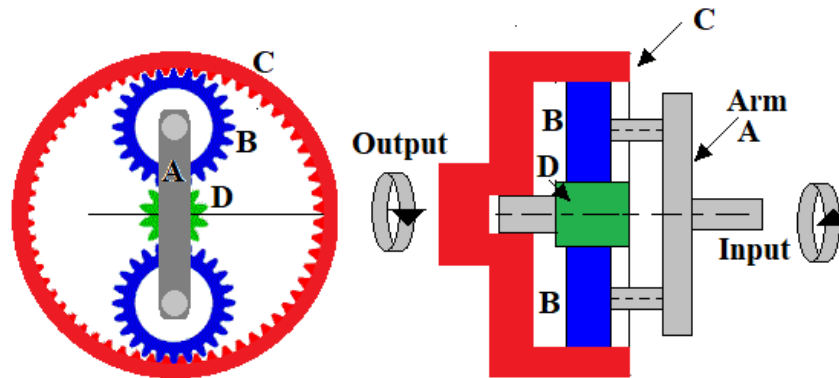$t_D = 240 - 40 = 200$

Identify that gear C is fixed and the arm must do one revolution so it must be rotated back one revolution holding the input stationary.

Step	Action	A	B	C	D
1	Revolve all once	1	1	1	1
2	Revolve C -1	0	-240/20	-1	240/200
3	Add	1	-11	0	2.2

The ratio A/D is then 1: 2.2 and this is the gear ratio.

METHOD 2

In this case the sun gear D is fixed and the internal gear C is made into the output.

WORKED EXAMPLE No. 6

An epicyclic gear box has a fixed sun gear D and the internal gear C is the output with 300 teeth. The planet gears B have 30 teeth. The input is the arm/cage A.

Calculate the number of teeth on the sun gear and the ratio of the gear box.

SOLUTION

$t_C = t_D + 2\, t_B$
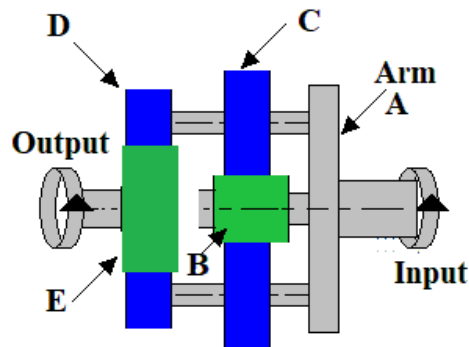$300 = t_D + 2 \times 30$
$t_D = 300 - 60 = 240$

Identify that gear D is fixed and the arm must do one revolution so it must be D that is rotated back one revolution holding the arm stationary.

Step	Action	A	B	C	D
1	Revolve all once	1	1	1	1
2	Revolve D -1	0	240/30	240/300	-1
3	Add	1	9	1.8	0

The ratio A/C is then 1: 1.8 and this is the gear ratio. Note that the solution would be the same if the input and output are reversed but the ratio would be 1.8.

METHOD 3

In this design a compound gear C and D is introduced. Gear B is fixed and gears C rotate upon it and around it. Gears C are rigidly attached to gears D and they all rotate at the same speed. Gears D mesh with the output gear E.

WORKED EXAMPLE No. 7

An epicyclic gear box is as shown above. Gear C has 100 teeth, B has 50, D has 50 and E has 100.

Calculate the ratio of the gear box.

SOLUTION

Identify that gear B is fixed and that A must do one revolution so it must be B that is rotated back one revolution holding A stationary.

| Step	Action	A	B	C/D	E
1	Revolve all once	1	1	1	1
2	Revolve B -1	0	-1	½	-¼
3	Add	1	0	1 ½	¾

The ratio A/E is then ¾ :1 or 3:4

Note that the input and output may be reversed but the solution would be the same with a ratio of 4:3 instead of 3:4

SELF ASSESSMENT EXERCISE No. 4

1. An epicyclic gear box is designed as shown. The input D rotates at 200 rev/min clockwise viewed from the left with a torque of 40 Nm. The efficiency is 75%.

    Calculate the following.

    i. The gear box ratio. (16:1)
    ii. The output speed and its direction. (12.5 rev/min clockwise)
    iii. The power output. (628.3 W)
    iv. The holding torque. (520 Nm anticlockwise)
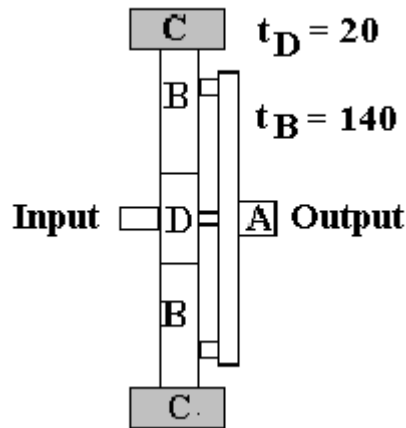
2. An epicyclic gear box is designed as shown. The input A rotates at 100 rev/min clockwise viewed from the right with a torque of 20 Nm. The efficiency is 65%.
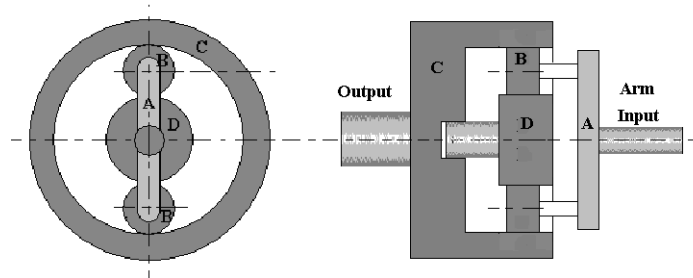
    Calculate the following.

    i. The gear box ratio. (1:5)
    ii. The output speed and its direction. (500 rev/min clockwise)
    iii. The power output. (136.1 W)
    iv. The holding torque. (22.6 Nm anticlockwise)
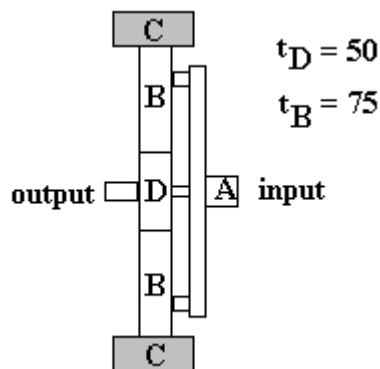
3. An epicyclic gear box as shown has a fixed sun gear D and the internal gear C is the output with 400 teeth. The planet gears B have 150 teeth. The input is the arm/cage A. The output must deliver 5 kW of power at 900 rev/min. The input power is 7 kW.

    Calculate the following.

    i. The gear box ratio. (4:5)
    ii. The input speed and its direction. (720 rev/min same direction as input)
    iii. The efficiency. (71.4%)
    iv. The holding torque. (145.9 Nm)

4. An epicyclic gear box as shown has a fixed sun gear B with 150 teeth. Gear C has 30 teeth and it is compounded with D which has 130 teeth. The input shaft delivers 200 W at 2400 rev/min. The gear box efficiency is 55%.

    Calculate the following.

    i. The number of teeth on E. (50)
    ii. The gear box ratio. (12:1)
    iii. The output speed and its direction. (200 rev/min in reverse direction)
    iv. The holding torque. (4.46 Nm)
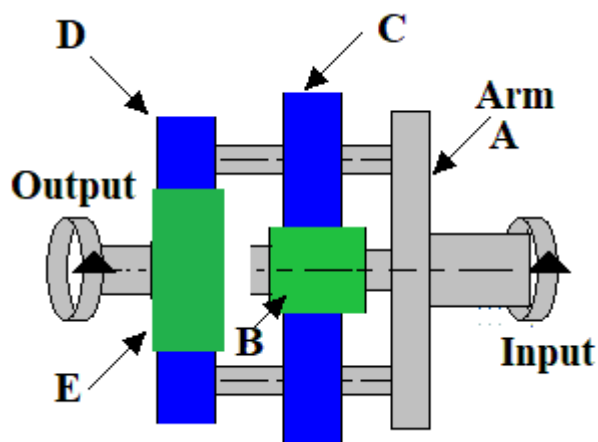
5. *Acceleration of Gear Trains*

Since all points on the circle must move without slipping, then not only must the velocity be the same but also the acceleration. The acceleration of a point on the circle is 'a' m/s^2. This is related to the angular acceleration α by

$$a = \frac{\alpha D}{2}$$

D is proportional to the teeth. Since a is the same for all then $\alpha_A t_A = \alpha_B t_B = \alpha_C t_C$

All bodies have mass (inertia) and so a force is needed to change their motion. In the case of a wheel, torque is needed to produce changes in the angular speed and Newton's law of motion for a wheel is

$$T_i = I \alpha$$

I is the moment of inertia of the wheel. The moment of inertia is found from $I = Mk^2$ where k is the radius of gyration of the wheel (the effective radius of the rotating mass). In real gear trains, friction and other loads placed on the gears produce extra torque which must be added to the inertia torque. Note the inertia torque is only produced when there are changes to the motion and not when running at constant speed. We will only consider a simple gear train here.

Simple Gear Train

Consider out simple gear train again. The power produced by a wheel is ωT.
If a torque T_C exists on the shaft of gear C, it must have originated from a torque on the shaft of the input gear (Gear A). Assuming no energy loss, we may determine T_A by equating input and output power.

Idler

$$\omega_A T_A = \omega_B T_B = \omega_C T_C$$
$$T_A = \omega_C T_C / \omega_A = T_C (t_A/t_C)............ (1)$$
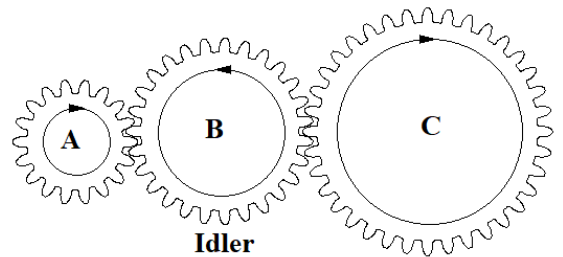
Similarly if a torque exists on the shaft of gear B, the Torque resulting on gear A is

$$T_A = T_B \times \frac{t_A}{t_B}$$

Consider that the torque is due to acceleration of the gear train (inertia torques).
Torque required to accelerate gear A is $= I_A \alpha_A$
Torque required to accelerate gear B is $= I_B \alpha_B$
Torque required to accelerate gear A is $= I_C \alpha_C$

All these torques must be provided by gear A. These are found by use of equation (1). The total torque on A is

$$T_A = I_A \alpha_A + I_B \alpha_B \times \frac{t_A}{t_B} + I_C \alpha_C \times \frac{t_A}{t_C}$$

Since the accelerations are related by $\alpha_A t_A = \alpha_B t_B = \alpha_C t_C$ we may convert all the accelerations into α_A. The torque becomes

$$T_A = I_A \alpha_A + I_B \alpha_A \left(\frac{t_A}{t_B}\right)^2 + I_C \alpha_A \left(\frac{t_A}{t_C}\right)^2 = \alpha_A \left[I_A + I_B \left(\frac{t_A}{t_B}\right)^2 + I_C \left(\frac{t_A}{t_C}\right)^2 \right]$$

$$\left[I_A + I_B \left(\frac{t_A}{t_B}\right)^2 + I_C \left(\frac{t_A}{t_C}\right)^2 \right]$$ is called the effective moment of inertia

WORKED EXAMPLE No. 8

Calculate the input torque required to accelerate the input gear A at 6.67 rad/s^2. There are no other loads or losses on the gears.

20 teeth
I = 1 kg m^2

80 teeth
I = 10 kg m^2

A   B
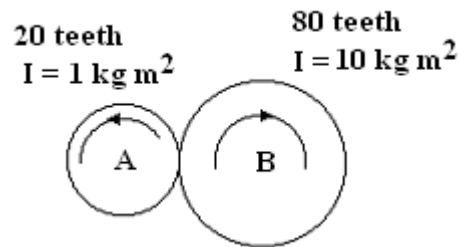
SOLUTION

$$I(\text{effective}) = \left[I_A + I_B \left(\frac{t_A}{t_B} \right)^2 + I_C \left(\frac{t_A}{t_C} \right)^2 \right] = 1 + 10 \left(\frac{10}{80} \right)^2 = 1.625 \text{ kg m}^2$$

$$T_A = \alpha_A \, I = 6.67 \times 1.625 = 10.8 \text{ Nm}$$

SELF ASSESSMENT EXERCISE No. 5

Calculate the input torque required to accelerate gear input A at 6.67 rad/s^2. (173.3 Nm)

80 teeth
I = 10 kg m^2

20 teeth
I = 1 kg m^2

A   B