SOLID MECHANICS

DYNAMICS

TUTORIAL – CENTRIPETAL FORCE

On completion of this tutorial you should be able to

- Explain and define centripetal and centrifugal acceleration.
- Explain and define centripetal and centrifugal force.
- Solve problems involving centripetal and centrifugal force.
- Derive formulae for the stress and strain induced in rotating bodies.
- Solve problems involving stress and strain in rotating bodies.
- Analyse problems involving vehicles skidding and overturning on bends.

Contents

1. Introduction

Centrifugal force is experienced by all rotating bodies and it is in fact the opposite and equal of centripetal force. Centripetal force is the force that pulls the body towards the centre of rotation in order to prevent it moving in a straight line. There are many ways to experience these forces on round-a-bouts, carousels, wall of death rides or simply riding a bike or travelling in a car around a bend. There are many mechanical devices that make good use of this such as clutches and speed governors (illustrated below). In this tutorial you will derive the equation for force and study some applications.

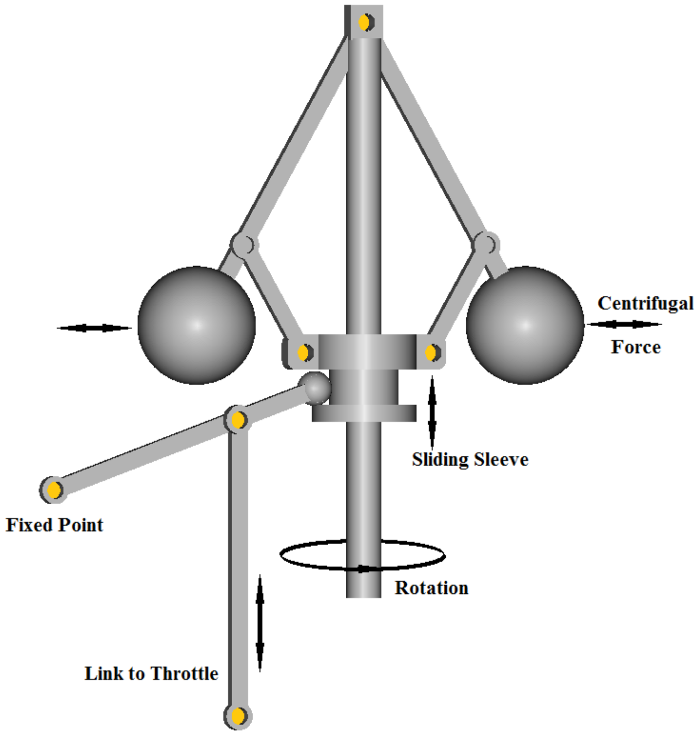

Figure 1

2. *Centripetal Affects*

2.1 Acceleration and Force

Centripetal acceleration occurs with all rotating bodies. Consider a point P rotating about a centre O with constant angular velocity ω (fig. 2).

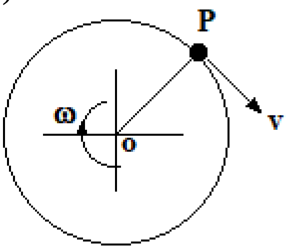

Figure 2

The radius is the length of the line O - P. The tangential velocity of P is v = ωR. This velocity is constant in magnitude but is continually changing direction.

$$a = \frac{\text{change in velocity}}{\text{time taken}} = \frac{\Delta v}{t}$$

Let's remind ourselves of the definition of acceleration.

Velocity is a vector quantity and a change in direction alone is sufficient to produce a change. It follows that a point travelling in a circle is continuously changing its direction, velocity and hence has acceleration.

Next let's remind ourselves of Newton's second Law of Motion which in its simplest form states

Force = Mass x acceleration.

It follows that anything with mass travelling in a circle must require a force to produce the acceleration just described.

The force required to make a body travel in a circular path is called ***Centripetal Force*** and it always pulls towards the centre of rotation. You can easily demonstrate this for yourself by whirling a small mass around on a piece of string.

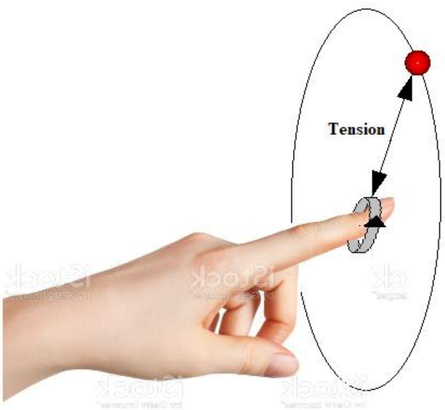

Figure 3

Let's remind ourselves of Newton's Third Law. **Every force has an equal and opposite reaction.**

The opposite and equal force of centripetal is the ***Centrifugal Force***.

The string is in tension and this means it pulls in both directions. The force pulling the ball towards the middle is the centripetal force and the force pulling on your finger is the centrifugal force.

The derivation of the formula for centripetal force and acceleration is done by considering the velocity as a vector.

Consider the velocity vector before and after point P has revolved a small angle $\delta\theta$. The magnitude of v_1 and v_2 are equal so let's denote it simply as v. The direction changes over a small period of time δt by $\delta\theta$ radians. We may deduce the change by using the vector addition rule.

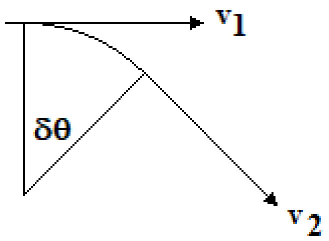

Figure 4

The first vector + the change = Final vector.

The rule is $v_1 + \delta v = v_2$. This is illustrated.

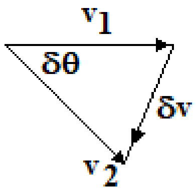

Figure 5

δv is almost the length of an arc of radius v. If the angle is small, this becomes truer.
The length of an arc is radius x angle so it follows that $\delta v = v\,\delta\theta$

This change takes place in a corresponding small time δt so the rate of change of velocity is:

$$\frac{\delta v}{\delta t} = \frac{v\delta\theta}{\delta t}$$

In the limit as $\delta t \rightarrow dt$

$$\frac{\delta v}{\delta t} \rightarrow \frac{dv}{dt} = \text{a (the acceleration)}$$

$$a = \frac{dv}{dt} = \frac{v\,d\theta}{dt} \quad \omega = \frac{d\theta}{dt} \quad \text{hence } \mathbf{a = v\,\omega}$$

Since $v = \omega R$ then substitute for v and $a = \omega^2 R$ and this is the centripetal acceleration.

Centripetal acceleration = ω^2R

Since $\omega = v/R$ then substitute for ω and $a = v^2/R$

Centripetal acceleration = v^2/R

If we examine the vector diagram, we see that as $\delta\theta$ becomes smaller and smaller, so the direction of δv becomes radial and inwards. The acceleration is in the direction of the change in velocity and so centripetal acceleration is radial and inwards.

If point P has a mass M, then the force required to accelerate this mass radial inwards is found from Newton's 2nd Law.

Centripetal force = M ω^2 R

Or in terms of velocity v

Centripetal force = M v^2/R

Centrifugal force is the reaction force and acts radial outwards.

WORKED EXAMPLE No. 1

Calculate the centripetal acceleration and force acting on an aeroplane of mass 1 500 kg turning on a circle 400 m radius at a velocity of 300 m/s.

SOLUTION

Centripetal acceleration = $v^2/R = 300^2/400 = 225$ m/s^2.

Centripetal force = mass x acceleration = $1\,500 \times 225 = 337.5$ kN

WORKED EXAMPLE No. 2

Calculate the centripetal force acting on a small mass of 0.5 kg rotating at 1 500 rev/minute on a radius of 300 mm.

SOLUTION

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1\,500}{60} = 157 \text{ rad/s}$$

Cent. acc. = $\omega^2 R = (157)^2 \times 0.3 = 7\,395$ m/s^2.

Cent. force = Mass $\times$ acc. = $0.5 \times 7\,395 = 3\,697$ N

WORKED EXAMPLE No. 3

A centrifugal clutch is shown in the diagram. The clutch must transmit a torque of 18 Nm at a speed of 142 rev/min. The coefficient of friction 'μ' between the drum and the friction lining is 0.3. The radius to the centre of gravity of each sliding head is 0.21 m and the inside radius of the drum is 0.25 m. Calculate the required mass of the sliding heads.

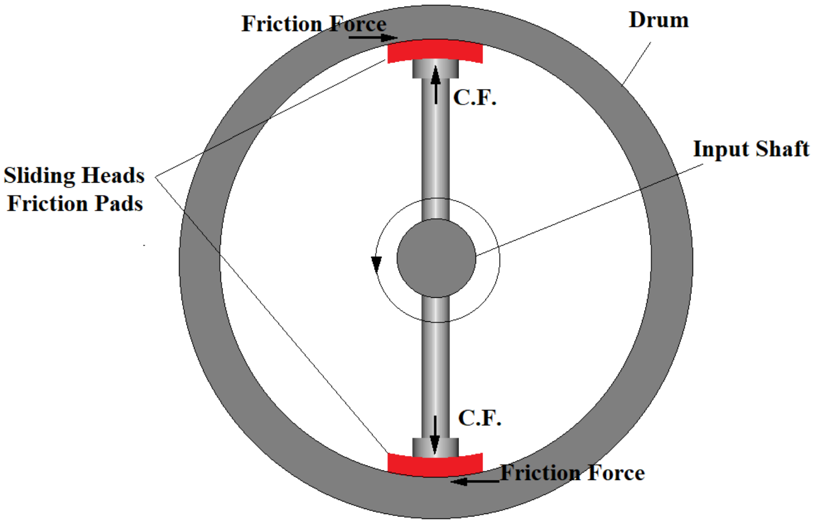

Figure 6

SOLUTION

Torque = 18 Nm, radius = 0.25 m T = Friction force x radius

Friction force = 18/0.25 = 72 N
This is divided between two friction pads so each must produce 72/2 = 36 N each.

From the law of friction, Friction force = $\mu \times$ normal force
Normal force = 36/0.3 = 120 N

The normal force acts in a radial direction and must be equal to the centripetal force.
Centripetal force = $M\omega^2 r$

Equating forces we have $M\omega^2 r = 120$

Speed = 142 rev/min or 142/60 rev/s

Radius to centre of gravity = 0.21 m

Angular velocity $\omega = 2\pi \times$ speed $= 2\pi \times 142/60 = 14.87$ rad/s

$M \times 14.87^2 \times 0.21 = 120$

$$M = \frac{120}{14.87^2 \times 0.21} = 2.58 \text{ kg for each sliding head}$$

SELF ASSESSMENT EXERCISE No. 1

1. A centrifugal clutch is similar to that shown in figure 6.
   The clutch must transmit a torque of 25 Nm. The coefficient of friction 'μ' between the drum and the friction lining is 0.4. Each sliding head has a mass of 0.4 kg acting at a radius of 0.15 m. The inside radius of the drum is 0.18 m. Calculate the minimum speed required.
   (Answer 53.78 rad/s or 513.7 rev/min)

2. A rotating arm has a sliding mass of 2 kg that normally rests in the position shown with the spring uncompressed. The mass is flung outwards as it revolves. Calculate the stiffness of the spring such that the mass compresses it by 20 mm when the arm revolves at 500 rev/min.
   (Answer 38.4 N/mm)

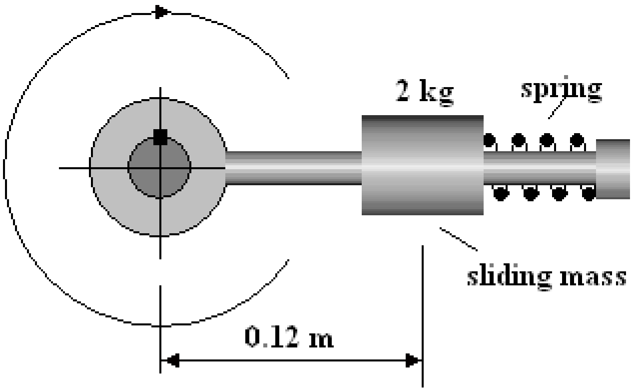

Figure 7

3 *Stress And Strain in Rotating Bodies*

3.1 *Stress*

Wheels experience stresses in them because the centripetal reaction (centrifugal force) tends to stretch the material along a radius. This can cause the wheel to disintegrate if it runs to fast. The effect might be catastrophic in a grinding wheel or flywheel.

Even if the wheel does not disintegrate, the stress will cause it to expand. The blades on a turbine or compressor rotor will stretch slightly under this strain and might touch the casing.

Consider a bar of uniform cross section A rotating about its centre as shown.

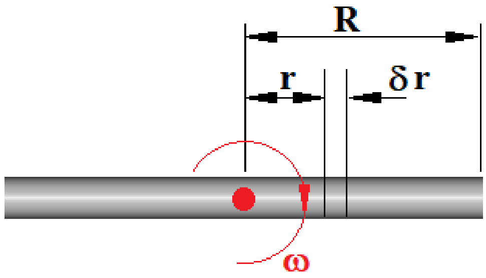

Figure 8

The radius to the tip is R. Consider a small length δr at radius r. The mass of this element is δm = ρAδr. The centrifugal force produced by this mass is δF.

$$\delta F = \delta m\, \omega^2 r = \rho\, A\, \delta r\, \omega^2 r$$

In the limit as $\delta r \to dr$ this becomes $dF = \rho\, A\, dr\, \omega^2 r$

The centrifugal force acting on this section due to the mass between r and R is then found by integration.

$$F = \rho A \omega^2 \int_r^R r\, dr = \rho A \omega^2 \frac{R^2 - r^2}{2}$$

The tensile stress induced at this section (at radius r) is the force per unit area.

$$\sigma = \frac{F}{A} = \rho A \omega^2 \frac{R^2 - r^2}{2A} = \rho \omega^2 \frac{R^2 - r^2}{2}$$

3.2 *Strain*

Consider the same element (figure 8). Suppose the element δr increases in length by δx. In the limit as $\delta r \to dr$, the strain in this short length is then

$$\varepsilon = \frac{dx}{dr} = \frac{\sigma}{E} = \frac{\rho \omega^2}{2E}\{R^2 - r^2\}$$

$$\varepsilon = \frac{\rho \omega^2}{2E}\{R^2 - r^2\}$$

The extension of the short length is

$$dx = \frac{\rho \omega^2}{2E}\{R^2 - r^2\}dr$$

$$x = \int_r^R \frac{\rho\omega^2}{2E}\{R^2 - r^2\}dr = \frac{\rho\omega^2}{2E}\int_r^R \{R^2 - r^2\}dr = \frac{\rho\omega^2}{2E}\left[R^2 r - \frac{r^3}{3}\right]_r^R$$

$$x = \frac{\rho\omega^2}{2E}\left[\left(R^3 - \frac{R^3}{3}\right) - \left(R^2 r - \frac{r^3}{3}\right)\right]$$

$$x = \frac{\rho\omega^2}{2E}\left[\frac{2R^3}{3} - R^2 r + \frac{r^3}{3}\right]$$

WORKED EXAMPLE No. 4

A bar 0.5 m long with a uniform section is revolved about its centre. The density of the material is 7 830 kg/m^3. The tensile stress in the material must not exceed 600 MPa. Calculate the speed of rotation that produces this stress. Go on to calculate the extension of the bar. The elastic modulus is 200 GPa.

SOLUTION

$$\sigma = 600 \times 10^6 = \rho\omega^2 \frac{R^2 - r^2}{2}$$

The maximum stress will be at the middle with r = 0 and R = 0.25 m

$$600 \times 10^6 = 7\,830\omega^2 \frac{[0.25^2 - 0^2]}{2}$$

$$\omega^2 = \frac{600 \times 10^6}{7\,830} \times \frac{2}{0.25^2} = 2.452 \times 10^6 \quad \omega = 1\,566 \text{ rad/s}$$

Converting to rev/min

$$N = \omega \times \frac{60}{2\pi} = 1\,566 \times \frac{60}{2\pi} = 14\,953 \text{ rev/min}$$

The extension of one half of the bar from r = 0 to R = 0.25 m is

$$x = \frac{\rho\omega^2}{2E}\left[\frac{2R^3}{3} - R^2 r + \frac{r^3}{3}\right] = \frac{7\,830 \times 1\,566^2}{2 \times 200 \times 10^9}\left[\frac{2 \times 0.25^3}{3} - 0 + 0\right]$$

x = 0.0005 m or 0.5 mm

SELF ASSESSMENT EXERCISE No. 2

1a. The blade of a turbine rotor has a uniform cross section. The root of the blade is attached to a hub of radius r. The tip of the blade has a radius R. Show that the stress at the root of the blade due to centrifugal force is given by

$$\sigma = \rho\omega^2 \frac{R^2 - r^2}{2}$$

1b. The hub radius is 200 mm and the tip radius is 320 mm.

The density of the blade material is 8 200 kg/m^3. Calculate the stress at the root when the rotor turns at 7 500 rev/minute.

(157.8 MPa)

1c. Derive the formula for the extension of the blade tip x given below.

$$x = \frac{\rho\omega^2}{2E}\left[\frac{2R^3}{3} - R^2 r + \frac{r^3}{3}\right]$$

where E is the elastic modulus.

1d. Calculate the extension of the tip given that E = 206 GPa.

(0.0495 mm)

4. Application to Vehicles

4.1 Horizontal Surface

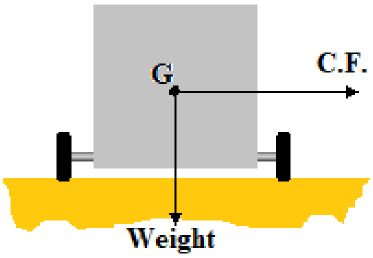

Figure 9

When a vehicle travels around a bend, it is subject to centrifugal force. This force always acts in a radial direction. If the bend is in a horizontal plane then the force always acts horizontally through the centre of gravity. The weight of the vehicle is a force that always acts vertically down through the centre of gravity. The diagram shows these two forces.

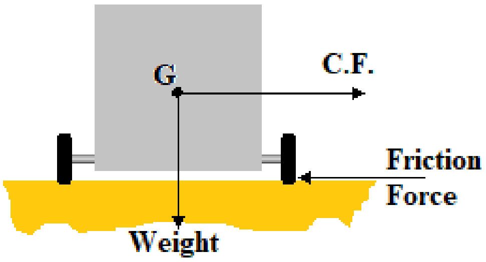

Figure 10

The centrifugal force tends to make the vehicle slide outwards. This is opposed by friction on the wheels. If the wheels were about to slide, the friction force would be μW where μ is the coefficient of friction between the wheel and the road. The tendency would be for the vehicle to overturn. If we consider the turning moments involved, we may solve the velocity which makes it overturn.

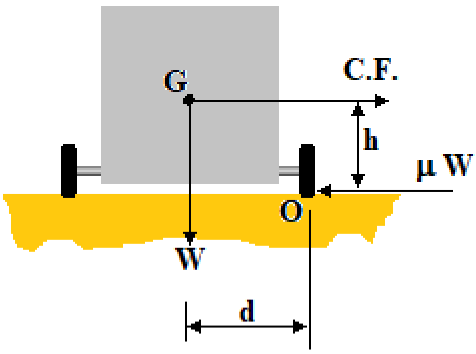

Figure 11

If the vehicle mass is M its weight is Mg. The radius of the bend is R.

Consider the turning moments about point O
The moment of force due to the weight is:

$$W\,d = M\,g\,d$$

The moment due to the centrifugal force is:

$$C.F.\ x\ h$$

Remember the formula for centrifugal force is:

$$CF = \frac{Mv^2}{R}$$

The moment due to the centrifugal force is:

$$C.F.\,x\,h = \frac{Mv^2}{R}\,x\,h$$

When the vehicle is about to overturn the moments are equal and opposite. Equating the moments about point O we get:

$$M = \frac{Mv^2}{R}\,x\,h = M\,g\,d$$

Rearrange to make v the subject.

$$v = \sqrt{\frac{g\,d\,R}{h}}$$

If the vehicle was about to slide sideways without overturning then the centrifugal force would be equal and opposite to the friction force. In this case:

$$\mu W = \mu Mg = \frac{Mv^2}{R} \text{hence } v = \sqrt{\mu g R}$$

Comparing the two equations it is apparent that it skids if **μ< d/h** and overturn if **μ>d/h**

WORKED EXAMPLE No. 5

A wheeled vehicle travels around a circular track of radius 50 m. The wheels are 2 m apart (sideways) and the centre of gravity is 0.8 m above the ground. The coefficient of friction is 0.4. Determine whether it overturns or slides sideways and determine the velocity at which it occurs

SOLUTION

Overturning

$$v = \sqrt{\frac{g\,d\,R}{h}} = \sqrt{\frac{9.81 \text{ x } 1 \text{ x } 50}{0.8}} = 24.76 \text{ m/s}$$

Sliding sideways

$$v = \sqrt{\mu\, g\, R} = \sqrt{0.4 \text{ x } 9.81 \text{ x } 50} = 14 \text{ m/s}$$

It follows that it will slide sideways when the velocity reaches 14 m/s.

4.2 Banked Surface

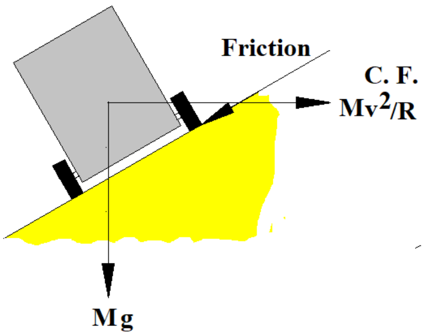

Figure 12

The road surface is flat and inclined at θ degrees to the horizontal but the bend is still in the horizontal plane. The analysis of the problem is helped if we consider the vehicle simply as block on an inclined plane as shown.

The C.F. still acts horizontally and the weight vertically. The essential distances required for moments about point O are the vertical and horizontal distances from O and these are h' and d' as shown.

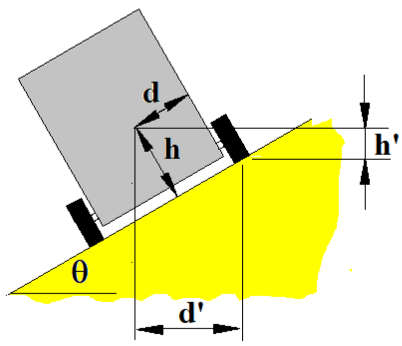

Figure 13

When the vehicle is just on the point of overturning the moments about the corner are equal and opposite as before. By applying trigonometry to the problem you should be able to show that

$h' = (h - d\tan\theta)\cos\theta$ $\quad\quad$ $d' = (h\tan\theta + d)\cos\theta$

Equate

$$\frac{Mv^2h'}{R} = Mgd' \qquad v^2h' = Rgd'$$

$$v^2h' = v^2(h - d\tan\theta)\cos\theta = Rg(h\tan\theta + d)\cos\theta$$

$$v^2 = \frac{Rg(h\tan\theta + d)\cos\theta}{(h - d\tan\theta)\cos\theta} = gR\left\{\frac{(h\tan\theta + d)}{(h - d\tan\theta)}\right\} = gR\left\{\frac{\left(\tan\theta + \frac{d}{h}\right)}{\left(1 - \frac{d}{h}\tan\theta\right)}\right\}$$

$$v^2 = gR\left\{\frac{\left(\frac{d}{h} + \tan\theta\right)}{\left(1 - \frac{d}{h}\tan\theta\right)}\right\}$$

This is the velocity at which the vehicle overturns.

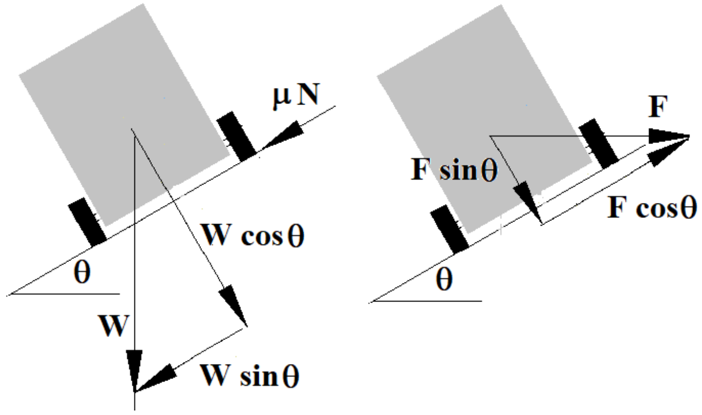

Figure 14

If the vehicle slides without overturning, then the total force **parallel** to the road surface must be equal to the friction force. Remember the friction force is μN where N is the total force acting normal to the road surface. Resolve all forces parallel and perpendicular to the road. The parallel forces are F cosθ and W sinθ as shown. The normal forces are F sinθ and W cosθ as shown. The friction force opposing sliding is μN

The forces acting parallel to the surface must be equal and opposite when sliding is about to occur.

Balancing all three forces we have $\mu N + W\sin\theta = F\cos\theta$

The total normal force N is the sum of the two normal forces. $N = W\cos\theta + F\sin\theta$

Substituting

$$\mu\{W\cos\theta + F\sin\theta\} + W\sin\theta = F\cos\theta$$

$$\mu W\cos\theta + \mu F\sin\theta + W\sin\theta = F\cos\theta$$

$$\mu Mg\cos\theta + \mu M\frac{v^2}{R}\sin\theta + Mg\sin\theta = M\frac{v^2}{R}\cos\theta$$

$$\mu g\cos\theta + \mu\frac{v^2}{R}\sin\theta + g\sin\theta = \frac{v^2}{R}\cos\theta$$

$$\mu\cos\theta + \mu\frac{v^2}{gR}\sin\theta + \sin\theta = \frac{v^2}{gR}\cos\theta$$

$$\mu + \mu\frac{v^2}{gR}\frac{\sin\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{v^2}{gR}$$

$$\mu + \mu\frac{v^2}{gR}\tan\theta + \tan\theta = \frac{v^2}{gR}$$

$$\tan\theta + \mu = \frac{v^2}{gR} - \mu\frac{v^2}{gR}\tan\theta = \frac{v^2}{gR}\{1 - \mu\tan\theta\}$$

$$\frac{\tan\theta + \mu}{1 - \mu\tan\theta} = \frac{v^2}{gR}$$

$$v^2 = gR\left\{\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right\}$$

This gives the velocity at which the vehicle slides.

WORKED EXAMPLE No. 6

A motor vehicle travels around a banked circular track of radius 80 m. The track is banked at 15^o to the horizontal. The coefficient of friction between the wheels and the road is 0.5. The wheel base is 2.4 m wide and the centre of gravity is 0.5 m from the surface measured normal to it. Determine the speed at which it overturns or skids.

SOLUTION

1. OVERTURNING

$$v^2 = gR\left\{\frac{\left(\frac{d}{h} + \tan\theta\right)}{\left(1 - \frac{d}{h}\tan\theta\right)}\right\} = 9.81 \text{ x } 80\left\{\frac{\left(\frac{1.2}{0.5} + \tan 15\right)}{\left(1 - \frac{1.2}{0.5}\tan 15\right)}\right\} = 76.6 \text{ m/s}$$

2. SKIDDING

$$v^2 = gR\left\{\frac{\mu + \tan\theta}{1 - \mu\tan\theta}\right\} = 9.81 \text{ x } 80\left\{\frac{0.5 + \tan 15}{1 - 0.5\tan 15}\right\} = 26.3 \text{ m/s}$$

The vehicle will skid before it overturns.

SELF ASSESSMENT EXERCISE No. 3

1. A motor vehicle travels around a banked circular track of radius 110 m. The track is banked at 8^o to the horizontal. The coefficient of friction between the wheels and the road is 0.4. The wheel base is 3 m wide and the centre of gravity is 0.6 m from the surface measured normal to it. Determine the speed at which it overturns or skids.
(66.3m/s to overturn and 24.9 m/s to skid)

2. A vehicle has a wheel base of 2.1 m and the centre of gravity is 1.1 m from the bottom. Calculate the radius of the smallest bend it can negotiate at 120 km/h. The coefficient of friction between the wheels and the track is 0.3.
(For skidding R = 377.5 m and for overturning R = 118.7 m hence the answer is 377.5 m)

3. Repeat question 3 given that the bend is banked at 10^o.

(For skidding R = 225.2 m and for overturning R = 83.3 m hence the answer is 225.2 m)