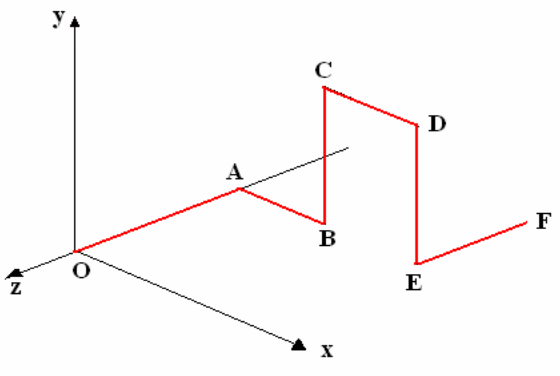
D225 2005 Q7

A steel rod 1600 mm long and 12 mm diameter is bent into the shape shown on the diagram. The density of steel is 7800 kg/m^3. The co-ordinates of the corners are given in the table below (in mm).

Point	x co-ordinate	y co-ordinate	z co-ordinate
O	0	0	0
A	0	0	-400
B	200	0	-400
C	200	300	-400
D	400	300	-400
E	400	0	-400
F	400	0	-600



a) Calculate I_{xx}, I_{yy}, I_{zz}, the moments of inertia of this object about the x, y and z axes respectively.

b) Supports are present at at O and A which allow the bent rod to rotate freely about the z axis and a torque of 1 Nm is applied to leg OA. Compute the angular acceleration of the object about the z axis and the reactions at O and A.

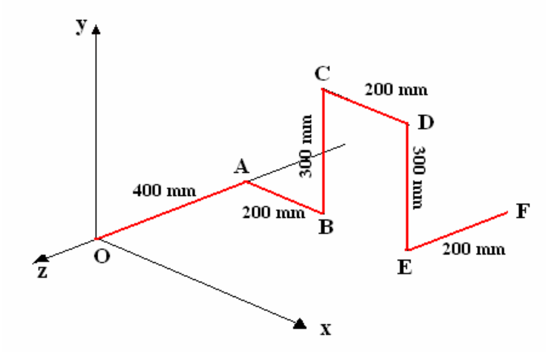
c) If the torque is sustained on the bent rod until it has rotated through exactly 10 full cycles, calculate the reaction forces at O and A at that instant.

SOLUTION

If there is a shorter way to find the moments of inertia and centre of gravity, then b) and c) are quite easy.

The moment of inertia of a uniform rod of length L is:
About the end $\rho AL^3/3$
About the middle $\rho AL^3/12$
The parallel axis theorem also gives $I = \{I_g + M k^2\}$ where k is the radius to the centre of gravity.

If $I_g = ML^2/12$ then $I = \rho AL\{L^2/12 + k^2\}$
$\rho A = 7800 \times \pi \times 0.12^2/4 = 0.8822$ kg/m



There may be a more systematic way to calculate the moment of inertia but it seems that we need to work out I_{xx}, I_{yy} and I_{zz} for each of the 6 legs - a lot of time needed.

SECTION OA L = 0.4

$$I_{xx} = \frac{\rho AL^3}{3} = 0.8822\left(\frac{0.4^3}{3}\right) = 0.018819\ \text{kg m}^2$$

$I_{yy} = I_{xx} = 0.18819$ kg m^2 $\quad I_{zz} = 0$

SECTION AB

$$I_{xx} = \rho AL\{0.4^2\} = 0.8822(0.2)\{0.4^2\} = 0.028229\ \text{kg m}^2$$

$$I_{yy} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.2)\left\{\frac{0.2^2}{12} + \left(0.4^2 + 0.1^2\right)\right\} = 0.03058\ \text{kg m}^2$$

$$I_{zz} = \frac{\rho AL^3}{3} = 0.8822\left(\frac{0.2^3}{3}\right) = 0.002352\ \text{kg m}^2$$

SECTION BC

$$I_{xx} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.3)\left\{\frac{0.3^2}{12} + \left(0.4^2 + 0.15^2\right)\right\} = 0.050283 \text{ kg m}^2$$

$$I_{yy} = Mk^2 = 0.8822(0.3)\left(0.2^2 + 0.4^2\right) = 0.050283 \text{ kg m}^2$$

$$I_{zz} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.3)\left\{\frac{0.3^2}{12} + \left(0.2^2 + 0.15^2\right)\right\} = 0.018525 \text{ kg m}^2$$

SECTION CD

$$I_{xx} = Mk^2 = 0.8822(0.2)\left(0.3^2 + 0.4^2\right) = 0.041108 \text{ kg m}^2$$

$$I_{yy} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.2)\left\{\frac{0.2^2}{12} + \left(0.4^2 + 0.3^2\right)\right\} = 0.044696 \text{ kg m}^2$$

$$I_{zz} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.2)\left\{\frac{0.2^2}{12} + \left(0.3^2 + 0.3^2\right)\right\} = 0.032347 \text{ kg m}^2$$

SECTION DE

$$I_{xx} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.3)\left\{\frac{0.3^2}{12} + \left(0.4^2 + 0.15^2\right)\right\} = 0.050283 \text{ kg m}^2$$

$$I_{yy} = \rho A\int_0^{0.3} r^2 dy \qquad r^2 = 0.4^2 + 0.4^2 = 0.32 \quad I_{yy} = 0.8822\int_0^{0.3}(0.32)dy = 0.8822\left[0.32y\right]_0^{0.3} = 0.084687 \text{ kg m}^2$$

$$I_{zz} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.3)\left\{\frac{0.3^2}{12} + \left(0.4^2 + 0.15^2\right)\right\} = 0.050283 \text{ kg m}^2$$

SECTION EF

$$I_{xx} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.2)\left\{\frac{0.2^2}{12} + \left(0.5^2\right)\right\} = 0.044696 \text{ kg m}^2$$

$$I_{yy} = \rho AL\left\{\frac{L^2}{12} + k^2\right\} = 0.8822(0.2)\left\{\frac{0.2^2}{12} + \left(0.5^2 + 0.4^2\right)\right\} = 0.072928 \text{ kg m}^2$$

$$I_{zz} = \rho AL\, k^2 = 0.8822(0.2)(0.4)^2 = 0.0282291 \text{ kg m}^2$$

TOTALS

	I_{xx}	I_{yy}	I_{zz}
OA	0.18819	0.18819	0
AB	0.02822	0.03058	0.00235
BC	0.05028	0.05028	0.01852
CD	0.04410	0.04469	0.03234
DE	0.05028	0.08468	0.05028
EF	0.04469	0.07292	0.02823
TOTAL	0.40576	0.04713	0.13172

There is much scope for error in arriving at these figures.

Part (b) Angular acceleration = $\alpha = T/I_{zz} = 1/0.13172 = 7.6$ rad/s^2

The centre of gravity is required and this requires another lengthy calculation. We need to find the centre of gravity in each plane.
In the z direction (r is the radius to the centre of each leg in the z direction)

Section	r	length L	mass = ρAL	m r
OA	0.2	0.4	0.35286	0.070572
AB	0.4	0.2	0.17643	0.070572
BC	0.4	0.3	0.26464	0.105856
CD	0.4	0.2	0.17643	0.0.070572
DE	0.4	0.3	0.26464	0.105856
EF	0.5	0.2	0.17643	0.088215
Totals			1.41143	0.511643

Check total mass = 1.6ρA = 1.4115
The centre of gravity is $\bar{z} = 0.511643/1.41143 = 0.3625$ m

In the x direction (r is the radius to the centre of each leg in the x direction)

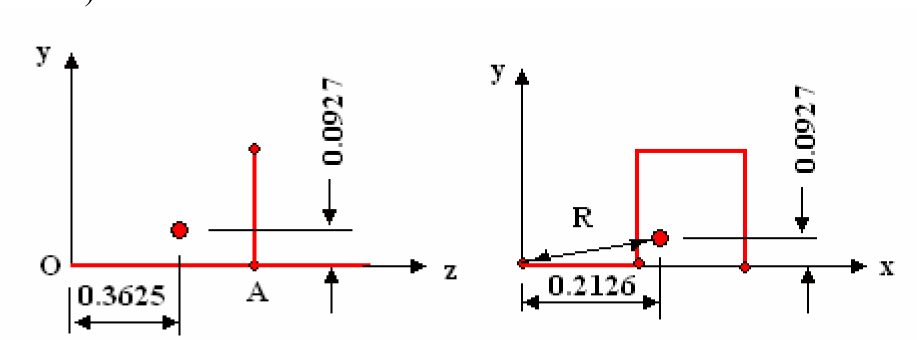
Section	r	length L	mass = ρAL	m r
OA	0	0.4	0.35286	0
AB	0.1	0.2	0.17643	0.017643
BC	0.2	0.3	0.26464	0.052928
CD	0.3	0.2	0.17643	0.052928
DE	0.4	0.3	0.26464	0.105856
EF	0.4	0.2	0.17643	0.070572
Totals			1.41143	0.3

The centre of gravity is $\bar{x} = 0.3/1.41143 = 0.2126$ m

In the y direction (r is the radius to the centre of each leg in the y direction)

Section	r	mass = ρAL	m r
OA	0	0.35286	0
AB	0	0.17643	0
BC	0.15	0.26464	0.039696
CD	0.3	0.17643	0.052928
DE	0.15	0.26464	0.039696
EF	0	0.17643	0
Totals		1.41143	0.13232

The centre of gravity is $\bar{y} = 0.13232/1.41143 = 0.0937$ m
The centre of gravity of the mass rotating about the z axis is at radius R as shown.
$R = (0.2126^2 + 0.0937^2)^{½} = 0.2323$



At the instant the torque is applied the angular velocity is zero so no centrifugal force is produced. If the body is free to rotate it will rest with the centre of gravity vertically below the z axis. Only the weight of the body acting at the centre of gravity need be considered. Taking moments about 'O' we have:

$Mg\ (0.3625) = R_A\ (0.4)$
$(1.4115)(9.81)\ (0.3625) = R_A\ (0.4)$
$13.8468\ (0.3625) = R_A\ (0.4)$
$R_A = 12.586$ N
$R_O = 13.8468 - 12.586 = 1.298$
I am not sure if this is what the examiner wants or whether weight is ignored in this problem.

Part (c)
After 10 revolutions an angular velocity ω is attained.
The centrifugal force is $M\omega^2R = 1.4115\ \omega^2\ 0.2323 = 0.3279\ \omega^2$
Angle rotated = $\theta = 10 \times 2\pi$ radian
$\omega^2 = (2\alpha\theta) = (2)(7.6)(20\pi) = 955$ N

Ignoring the weight and again taking moments about 'O' we have:

$(955)(0.3625) = R_A\ (0.4)$
$R_A = 865.5$ N
$R_O = 955 - 865.5 = 89.5$ N