The diagram shows a two mass system with details.

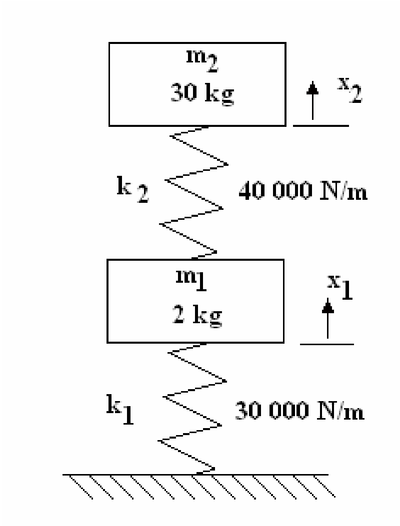
(a) Prepare appropriate mass and stiffness matrices for this system making clear what displacement co-ordinates have been chosen.

(b) Determine the natural frequencies of the system.

(c) For the lowest natural frequency, calculate the corresponding mode shape.

(d) Use the principle of mass orthoganality to determine the other mode shape.

(e) If sinusoidal forcing function is applied to the upper mass (the 30 kg mass), it is found that at one particular frequency the motion of that mass is zero. Calculate that frequency.



SOLUTION

(a)

Balancing forces on m_1 we have $m_1\dot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0$

Balancing forces on m_2 we have $m_2\dot{x}_{21} + k_2(x_2 - x_1) = 0$

The mass matrix is $\mathsf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}$

The stiffness matrix is $\mathsf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 70000 & -40000 \\ -40000 & 40000 \end{bmatrix}$

x is the distance moved from the rest position.

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 70000 & -40000 \\ -40000 & 40000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

To avoid trivial solutions we must have $\det\left[-\omega^2.\mathsf{M} + \mathsf{K}\right] = 0$

$$\det\left\{-\omega^2.\begin{bmatrix} 2 & 0 \\ 0 & 30 \end{bmatrix} + \begin{bmatrix} 70000 & -40000 \\ -40000 & 40000 \end{bmatrix}\right\} = 0$$

$$\det\begin{bmatrix} -2\omega^2 + 70000 & -40000 \\ -40000 & -30\omega^2 + 40000 \end{bmatrix} = 0$$

$$\left(-2\omega^2 + 70000\right)\left(-30\omega^2 + 40000\right) - \left(-40000\right)\left(-40000\right) = 0$$

$$60\omega^4 - 80\text{x}10^3\omega^2 - 2.1\text{x}10^6\omega^2 + 2.8\text{x}10^9 - 1.6\text{x}10^9 = 0$$

$$60\omega^4 - 2.18\text{x}10^6\omega^2 + 1.2\text{x}10^9 = 0$$

Solving the quadratic for ω^2 we get $\omega^2 = 35.77 \text{ x } 10^3$ and 560

$\omega = 189.1$ rad/s and 23.6 rad/s

Check answers with the derived formulae

$$\omega^2 = \frac{1}{2}\left(\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$\omega^2 = \frac{1}{2}\left(\frac{40000}{30} + \frac{70000}{2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{40000}{30} + \frac{70000}{2}\right)^2 - \frac{1200000000}{60}} = 18167 \pm 17608$$

$$\omega^2 = 35774 \text{ and } 558.7$$

$$\omega = 189.1 \text{ and } 23.63 \text{ rad/s}$$

(c) I haven't come across mass orthoganality but the other mode shape is:

Mode shapes are found by substituting the frequency into $[-\omega^2 M + K]X = 0$ where X is the mode shape vector.

$$\begin{bmatrix} -2\omega^2 + 70000 & -40000 \\ -40000 & -30\omega^2 + 40000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \text{ put } \omega = 23.6$$

$$\begin{bmatrix} 6886 & -40000 \\ -40000 & 23290 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$6886x_{11} - 4000x_{12} = 0$

$-4000x_1 + 23290x_2 = 0$

These are not simultaneous equations and we solve the ration x_2 by putting $x_1 = 1$

$6886 - 4000x_2 = 0$ hence $x_2 = 6886/4000 = 1.722$

$X_1 = \begin{bmatrix} 1 \\ 1.722 \end{bmatrix}$ for the lowest frequency.

(d) The other mode shape is $\omega = 189.1$

$$\begin{bmatrix} -2\omega^2 + 70000 & -40000 \\ -40000 & -30\omega^2 + 40000 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \text{ put } \omega = 23.6$$

$$\begin{bmatrix} 1518 & -40000 \\ -40000 & -1032764 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$1518x_{11} - 4000x_{12} = 0$

$-4000x_1 - 1032764x_2 = 0$

These are not simultaneous equations and we solve the ration x_2 by putting $x_1 = 1$

$1518 - 4000x_2 = 0$ hence $x_2 = 0.3795$

(e) Without proof (see tutorials on 2 DOF systems) the forcing frequency must then correspond to $\omega = \sqrt{\frac{k_1 + k_2}{m_1}} = \sqrt{\frac{70000}{2}} = 187 \text{ rad/s}$