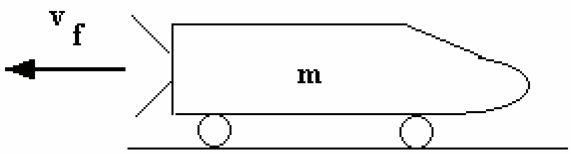
D225 1999 Q9

The diagram shows a rocket propelled vehicle. Fluid is ejected from the nozzle at a rate of dm_f/dt with a velocity of v_f relative to the vehicle.

(a) Show from first principles that the thrust produced is $F = v_f\, dm_f/dt$

(b) Show that the speed reached after T seconds is $v = v_f \ln(m_o/m)$ where mo is the initial mass of the vehicle and propellant and m is the final mass of the vehicle and propellant.

(c) The vehicle is propelled by ejecting water at a velocity of 100 m/s relative to the vehicle. The initial mass of the vehicle is 100kg not including the water. Calculate the mass of water needed to make the vehicle reach a speed of 30 m/s if all of it must be used up. Neglect any other forces opposing motion.



SOLUTION

(a) Newton's 2nd Law Impulse = rate of change of momentum.

$F\,\delta t = \delta(m v) \qquad F = \delta(m v)/\delta t$

Applied relative to the vehicle v is the constant velocity of the fuel (v_f) so F is the force produced by the fuel and $F = v_f\, dm_f/dt$. The vehicle is propelled by reaction to this force.

(b) $F = v_f \dfrac{dm_f}{dt}$ and this accelerates the vehicle by the law $F = m\,a$

'm' is mass of vehicle and 'a' is the acceleration. The mass of the vehicle reduces with time so

$$m = m_o - \int_0^t \frac{dm_f}{dt}\,dt$$

$$F = \left[m_o - \int_0^t \frac{dm_f}{dt}\,dt\right]a = \left[m_o - \int_0^t \dot{m}\,dt\right]a = \left[m_o - \dot{m}\,t\right]a$$

$$F = \left[m_o - \dot{m}\,t\right]a = v_f \frac{dm_f}{dt} = v_f\,\dot{m} \qquad a = \frac{v_f\,\dot{m}}{m_o - \dot{m}\,t}$$

This gives the acceleration of the vehicle at any time t when the propellant is ejected at a rate of $\dot{m}$. The velocity attained by the vehicle is found as follows.

$$v = \int_0^T a\,dt = \int_0^T \frac{v_f\,\dot{m}}{m_o - \dot{m}\,t}\,dt = v_f\,\dot{m}\int_0^T \frac{dt}{m_o - \dot{m}\,t}$$

$$\text{let } x = m_o - \dot{m}\,t \qquad dx = -\,\dot{m}\,dt \qquad dt = \frac{-\,dx}{\dot{m}}$$

$$v = v_f\,\dot{m}\int_0^T \frac{-\,dx}{\dot{m}x} = v_f\left[-\ln x\right]_0^T \qquad v = v_f\left[-\ln\left(m_o - \dot{m}\,t\right)\right]_0^T$$

$$v = \dot{m}\left[-\ln\left\{\frac{m_o - \dot{m}\,T}{m_o}\right\}\right] = \dot{m}\left[\ln\left\{\frac{m_o}{m_o - \dot{m}T}\right\}\right]$$

The final mass of the vehicle is $m = mo - \dot{m}T$ $v = v_f\left[\ln\left\{\dfrac{m_o}{m}\right\}\right]$

(c) $m_o = 100 + m_w$ m = 100 kg if all the water is used up. m_w = initial mass of water. v_f = 100 m/s

$$30 = 100\left[\ln\left\{\frac{100 + m_w}{100}\right\}\right] \qquad e^{0.3} = \frac{100 + m_w}{100} = 1.35$$

$$135 = 100 + m_w \qquad m_w = 35\,\text{kg}$$