ENGINEERING COUNCIL EXAM MATHEMATICS C101
SOLUTIONS TO EXAM PAPER 2004

Q7a

Show that the following system of linear equations does not possess a unique solution, and find a parametric means of describing the family of solutions.

$5x_1 + 8x_2 - 5x_3 = 2$
$4x_1 + 5x_2 - 3x_3 = 0$
$3x_1 + 2x_2 - x_3 = -2$

SOLUTION

In matrix form we have

$$\begin{bmatrix} 5 & 8 & -5 \\ 4 & 5 & -3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

If the determinant of the coefficient matrix is zero there is no unique solution. Find the determinant.

Find the minors

$$M_{11} = \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix} = (5)(-1) - (-3)(2) = -5 + 6 = 1$$

$$M_{12} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \end{bmatrix} = (4)(-1) - (-3)(3) = -4 + 9 = 5$$

$$M_{13} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} = (4)(2) - (5)(3) = 8 - 15 = -7$$

The cofactors go + - + so
$A_{11} = 1$ $A_{12} = -5$ $A_{13} = -7$

From the matrix $a_{11} = 5$ $a_{12} = 8$ $a_{13} = -5$

The determinant is $a_{11} A_{11} + a_{12}A_{12} + a_{13} A_{13} = (5)(1) + (8)(-5) + (-5)(-7) = 5 - 40 + 35 = 0$

The determinant is zero so there is no unique solution.

Q7b
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

For each eigenvalue find a suitable eigenvector and show that the three vectors are mutually perpendicular.

SOLUTION

$$A - \lambda I = A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

We must find the determinant of this matrix.
Find the minors

$$M_{11} = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - (0)(0) = 2 - 3\lambda + \lambda^2$$

$$M_{12} = \begin{bmatrix} 0 & 0 \\ 2 & 1-\lambda \end{bmatrix} = (0)(1-\lambda) - (0)(2) = 0$$

$$M_{13} = \begin{bmatrix} 0 & 2-\lambda \\ 2 & 0 \end{bmatrix} = (0)(0) - (2-\lambda)(2) = -4 + 2\lambda$$

The cofactors go + - + so
$A_{11} = 2 - 3\lambda + \lambda^2$ $\quad A_{12} = 0$ $\quad A_{13} = -4 + 2\lambda$

From the matrix $a_{11} = 1-\lambda$ $\quad a_{12} = 0$ $\quad a_{13} = 2$

The determinant is $a_{11} A_{11} + a_{12}A_{12} + a_{13} A_{13} = (1-\lambda)(2 - 3\lambda + \lambda^2) + 0 + 2(-4 + 2\lambda)$
$0 = -\lambda^3 + 4\lambda^2 - \lambda - 6$
$0 = \lambda^3 - 4\lambda^2 + \lambda + 6$
$0 = (\lambda - 2)(\lambda + 1)(\lambda - 3)$ The eigenvalues are 2, -1 and 3

Find the eigenvectors
$(A - \lambda I)X = 0$ where X is the eigenvector matrix.

First find the eigenvector for $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 1-2 & 0 & 2 \\ 0 & 2-2 & 0 \\ 2 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$(A - \lambda I)X = 0 = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We are solving three simultaneous equations. One possible solution is a = b = b = 0 but an eigenvector cannot be the zero vector.
The other two non zero equations indicate b = 0 and $-a + 2c = 0$ and $2a - c = 0$
The only vector that makes sense is for a = c = 0 and b to have any value so the vector is:
(0, b, 0) or $\alpha(0, 1, 0)$ where α is any scalar value.

Next find the eigenvector for λ = -1

$$(A - \lambda I)\,X = 0 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again, one possible solution is a = b = c = 0 but an eigenvector cannot be the zero vector.
The only vector that makes sense is for a = - c and b = 0
The eigenvector is (a, 0, -a) or β(1, 0 , -1) where β is any scalar value.

Next find the eigenvector for λ = 3

$$(A - \lambda I)\,X = 0 = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again, one possible solution is a = b = c = 0 but an eigenvector cannot be the zero vector.
The only vector that makes sense is for a = c and b = 0
The eigenvector is (a, 0, a) or γ(1, 0 , 1) where γ is any scalar value.

The three eigenvectors in Cartesian form are (0, y, 0) , (x ,0, -z) and (x, 0, z). These are all at 90^o to each other which cab be seen when sketched.