ENGINEERING COUNCIL EXAM MATHEMATICS C101
Solutions to exam paper 2004

Q1a. The deflection y of a beam of length L at a distance x from the its left hand end is given by :
$y = \alpha x^2(L-x)^2$ where α is a positive constant.

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and hence obtain the maximum deflection of the beam. Find also where the points of inflection occur.

SOLUTION

Examining the equation $y = \alpha x^2(L-x)^2$ we see that y is zero when x = 0 and when x = L so it appears to be a simply supported beam. The deflection appears to be upwards but note in beam problems we usually have a minus in the equation to indicate downwards deflection. As we have an exact equation we do not need constants of integration.

$$y = \alpha x^2(L-x)^2 = \alpha x^2\{L^2 - 2Lx + x^2\} = \alpha\{x^2L^2 - 2x^3L + x^4\}$$

$$\frac{dy}{dx} = \alpha\left[2xL^2 - 6x^2L + 4x^3\right] \qquad \frac{d^2x}{dy^2} = \alpha\left[2L^2 - 12xL + 12x^2\right]$$

For a maximum or minimum point $\frac{dy}{dx} = 0$

$0 = \left[2xL^2 - 6x^2L + 4x^3\right]$ A possible value of x is zero

Divide everything by x $0 = \left[2L^2 - 6xL + 4x^2\right]$ and factorise $0 = (2x-2L)(2x-L)$

Other possible solutions are x = L and x = L/2. Now check if these are maximum or minimum values by checking if the second derivative is positive or negative at these points.

Put x = 0 and $\frac{d^2x}{dy^2} = \alpha\left[2L^2\right]$ This is positive so it is probably a minimum

Put x = L $\frac{d^2x}{dy^2} = \alpha\left[2L^2 - 12L^2 + 12L^2\right] = \alpha\left[2L^2\right]$ This is positive so it is a probably a minimum.

Put x = L/2 $\frac{d^2x}{dy^2} = \alpha\left[2L^2 - 6L^2 + 3L^2\right] = -\alpha L^2$ This is negative so it is probably a maximum.

At a maximum the gradient changes from positive to negative so to be sure we should check.

$$\frac{dy}{dx} = \alpha\left[2xL^2 - 6x^2L + 4x^3\right]$$

Put x = 0.45L and $\frac{dy}{dx} = \alpha\left[0.9L^3 - 1.215L^3 + 0.3645L^3\right] = 0.0495\alpha$

Put x = 0.55L and $\frac{dy}{dx} = \alpha\left[1.1L^3 - 1.815L^3 + 0.1663L^3\right] = -0.55\alpha$

This confirms at x = L/2 the deflection is a maximum so putting x = L/2 into $y = \alpha\{x^2L^2 - 2x^3L + x^4\}$ we have $y_{\max} = \alpha\left\{\frac{L^4}{4} - \frac{L^4}{4} + \frac{L^4}{16}\right\} = \frac{\alpha L^4}{16}$

When we have a point of inflection, $\frac{d^2x}{dy^2} = 0$ so we must find these.

$\frac{d^2x}{dy^2} = \alpha\left[2L^2 - 12xL + 12x^2\right] = 0$ $\quad 6x^2 - 6xL + L^2 = 0$ This can only be solved by the quadratic formula. Coefficients a = 6, b = -6L. c = L^2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6L \pm \sqrt{36L^2 - 24L^2}}{12} = \frac{6L \pm \sqrt{12L^2}}{12} = 0.5L \pm \frac{L}{\sqrt{12}} = 0.7886\text{L and } 0.2113\text{L}$$

Substitute back and the answers check out.

Q1b The current i at time t in a series LR circuit is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$

To investigate the effect on the current as R becomes small, use L'Hospital's rule to derive an expression for i when R approaches zero. Verify the result by expanding i as a Maclaurin's series in powers of R

SOLUTION

Clearly if we put R = 0 $i = \frac{E}{0}\left(1 - e^{-0}\right) = \frac{E}{0}(1-1) = \frac{0}{0}$ and this is indeterminate.

Using L'Hospital's rule we must differentiate the top and bottom with respect to R

First rearrange to make it clear. $i = E\frac{\left(1 - e^{-\frac{Rt}{L}}\right)}{R}$ Then differentiate the top and bottom

$$\underset{R\to 0}{i} = E\left[\frac{\frac{t}{L}e^{-\frac{Rt}{L}}}{1}\right] = E\frac{t}{L}e^{-0} = \frac{Et}{L}$$

Maclaurin Expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots\dots$$

$$f(R) = f(0) + Rf'(0) + \frac{R^2}{2!}f''(0) + \dots\dots\dots$$

Expanding $i = e^{-\frac{Rt}{L}}$

$$f0) = 1 - \left(\frac{t}{L}\right)R + \left(\frac{t}{L}\right)^2\frac{R^2}{2} - \left(\frac{t}{L}\right)^3\frac{R^3}{6} + \dots\dots$$

Hence $i = \frac{E}{R}\left(1 - \left\{1 - \left(\frac{t}{L}\right)R + \left(\frac{t}{L}\right)^2\frac{R^2}{2} - \left(\frac{t}{L}\right)^3\frac{R^3}{6} + \dots\dots\right\}\right)$

$$i = \frac{E}{R}\left(1 - 1 + \left(\frac{t}{L}\right)R - \left(\frac{t}{L}\right)^2\frac{R^2}{2} + \left(\frac{t}{L}\right)^3\frac{R^3}{6} + \dots\dots\right)$$

$$i = \frac{E}{R}\left(\left(\frac{t}{L}\right)R - \left(\frac{t}{L}\right)^2\frac{R^2}{2} + \left(\frac{t}{L}\right)^3\frac{R^3}{6} + \dots\dots\right)$$

$$i = E\left(\frac{t}{L}\right)\left\{1 - \left(\frac{t}{L}\right)\frac{R}{2} + \left(\frac{t}{L}\right)\frac{R^2}{6} + \dots\dots\right\}$$

When R = 0

$i = E\frac{t}{L}$ giving the same result